Destabilizing Effects of Market Size in the Dynamics of Innovation*

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Abstract: In existing models of endogenous innovation cycles, market size alters the amplitude of fluctuations without changing the nature of fluctuations. This is due to the ubiquitous assumption of CES homothetic demand system, implying that monopolistically competitive firms sell their products at an exogenous markup rate in spite of the empirical evidence for the procompetitive effect of entry and market size. We extend the Judd model of endogenous innovation cycles to allow for the procompetitive effect, using a more general homothetic demand system. We show that a larger market size/innovation cost ratio, by reducing the markup rate through the procompetitive effect, has destabilizing effects on the dynamics of innovation under two complementary sets of sufficient conditions; i) when the price elasticity is "not too convex" in price; and ii) when the demand system belongs to the two parametric families, "generalized translog" and "constant pass-through," each of which features the choke price and yet contains CES as a limit case. Interestingly, the destabilizing effects become *amplified* as the demand system approaches to the CES limit within each family. We also discuss some cross-sectional implications in a multi-market extension. Because innovation/entry activities fluctuate more in larger markets, they are not always higher in larger markets than smaller markets. Furthermore, the sale of each product is more volatile in larger markets.

Keywords: Dynamic monopolistic competition, Endogenous innovation cycles, the Judd model, H.S.A., Procompetitive effect, Market size and volatility, Piecewise-linear dynamical system, Periodic cycle, Robust chaotic attractor
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1. Introduction

How does market size affect the dynamics of innovation? Many existing studies have already investigated the market size effect on innovation and long run growth.[1] However, innovation is not only a source of long run growth. It is also a source of fluctuations because innovations tend to arrive in waves, as many have pointed out.[2] Yet, little is known about the market size effect on the patterns of fluctuations in innovation and aggregate dynamics. In existing models of *endogenous innovation cycles*, market size merely affects the amplitude of fluctuations. Its potential effects on the patterns of fluctuations are muted by the ubiquitous assumption of the CES homothetic demand system for innovated products, which implies that monopolistically competitive firms sell their products at an exogenously constant markup rate, in spite of the empirical evidence of *the procompetitive effect*; see, e.g., Campbell and Hopenhayn (2005) and Feenstra and Weinstein (2017). That is, as more firms enter and compete against one another in a larger economy, they face more elastic demand for their products, which forces them to set their prices at lower markup rates. In the presence of such procompetitive effect, a larger market size relative to the innovation cost (or equivalently a smaller innovation cost relative to market size) and the resulting competitive pressures would make innovators more sensitive to changing market environments, thereby causing instability in the dynamics of innovation.

To capture this intuition, we extend the Judd (1985, section 4) model of endogenous innovation cycles to allow for the procompetitive effect. The Judd model offers an ideal setting for our purpose. First, it generates endogenous fluctuations along the unique equilibrium trajectory, unlike some other models of endogenous innovation cycles, which rely on expectational indeterminacy and multiple equilibria. Second, it is analytically tractable. Starting from any initial condition, its unique equilibrium trajectory can be obtained by iterating a skewed-V map (i.e., piecewise linear with two branches, decreasing in the lower branch and increasing in the upper branch). This class of maps generates a wide range of fluctuating patterns, including chaotic fluctuations, and yet it is simple enough to be characterized completely. In particular, one could study its properties by looking at a single constant number,

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[1] See, e.g., Romer (1990), Rivera-Batiz and Romer (1991), and Grossman and Helpman (1993). Acemoglu (2008), Aghion and Howitt (2008) and Gancia and Zilibotti (2005, 2009) offer more textbook treatments.

[2] The literature of endogenous innovation cycles, which captures this idea, includes Benhabib (2014), Deneckere and Judd (1992), Evans, Honkaponja and Romer (1998), Francois and Lloyd-Ellis (2003), Gale (1996), Gardini, Sushko, and Naimzada (2008), Iong and Irmen (2021), Jovanovic (2006), Jovanovic and Rob (1990), Judd (1985), Matsuyama (1999, 2001), Shleifer (1986), Stein (1997), and Wälde (2005).

θ, which we call the *delayed impact of innovation.* This constant number determines how much past innovations discourage current innovations; it is the key factor that generates incentives for innovators to synchronize their activities and creates temporal clustering of innovation in the model. Under CES, this number is a monotonically decreasing transformation of the (exogenously determined) constant markup rate and hence it is independent of any other parameters. Yet, it captures the idea that greater competitive pressures lead to instability. This feature makes it possible to generate the destabilizing effects of market size by endogenizing the markup rate through the procompetitive effect of market size.

We generalize the Judd model by extending its CES homothetic demand system to a more general homothetic demand system, H.S.A., which stands for *Homothetic with a Single Aggregator.* It is one of the classes of homothetic demand systems studied in Matsuyama and Ushchev (2017), to which we further impose symmetry and gross substitutability and define over a continuum of input varieties to make it applicable to monopolistic competition, as in Matsuyama and Ushchev (2020a, section 3). The key feature of monopolistic competition under H.S.A. is that the price elasticity of demand curve for each product is a function of its "relative price," which is defined as its own price divided by the price aggregator, which is *common* across all products. This common price aggregator fully captures the cross-price effects in the demand system, thus competitive pressures each innovator faces.

We have chosen this class of demand systems for the following reasons. First, they are *homothetic*. Although there have been many attempts to develop monopolistic competition models without CES, they have typically done so by making the demand system nonhomothetic.[3] In order to *isolate* the procompetitive effect of a market size change, it is useful to avoid introducing the market size effect operating through nonhomotheticity.[4] Furthermore, homotheticity makes it straightforward to extend it to multi-sector or multi-market settings.

[3]For example, Dixit and Stiglitz (1977, Section II) extended their monopolistic competition model to a class of non-CES demand systems, which have been further explored by Behrens and Murata (2007), Zhelobodko, Kokovin, Parenti, and Thisse (2012), Dhingra and Morrow (2019), Latzer, Matsuyama, and Parenti (2019), among others. Although Dixit and Stiglitz called this class, "Variable Elasticity Case," the well-known Bergson's Law states that, within the class of demand systems they considered, they are homothetic if and only if they are CES. In other words, any departure from CES within this class introduces nonhomotheticity. See Parenti, Thisse, and Ushchev (2017) and Thisse and Ushchev (2018) for more discussions on this issue with extensive references.
[4]Indeed, how to measure market size is far from obvious under nonhomotheticity. For example, a change in the aggregate expenditure would generally have different effects, depending on whether it is caused by a change in the population or by a change in per capita expenditure. Homotheticity allows us to abstract from such demand composition effects.

Second, under the additional assumption that the price elasticity function is increasing (i.e., the price elasticity goes up as one moves up along the demand curve; the so-called Marshall's second law of demand), H.S.A. exhibits the procompetitive effect.[5] Third, H.S.A. contains as special cases both CES and homothetic translog, which has been used to introduce the procompetitive effect.[6] Thus, H.S.A. allows us to perform robustness checks for these two demand systems. Fourth, as the name suggests, H.S.A. features a single aggregator, which serves as the sufficient statistic to capture all *the competitive pressures* each firm faces caused whether by entry of new firms or by pricing of other firms. Due to this single aggregator property, the Judd model under H.S.A. remains equally tractable as the original Judd model under CES. Indeed, its dynamics are still characterized by a skewed-V map.[7] The only difference from the case of CES is that both the markup rate as well as the delayed impact of innovation, θ, become functions of the market size/innovation cost ratio. Thus, by investigating the properties of these functions, we can use the Judd model under H.S.A. as a simple way of studying how the market size/innovation cost ratio affects the patterns of fluctuations in innovation dynamics through its procompetitive effect.[8]

In our analysis, we identify two complementary sets of sufficient conditions under which an increase in the market size/innovation cost ratio increases the delayed impact of innovation, θ, through the procompetitive effect, and hence it has the destabilizing effects on the dynamics of innovation. The first set of the sufficient conditions is that the price elasticity as a function of its relative price, is not "too convex," that is, the price elasticity goes up when moving up along the demand curve, but not in a too accelerating way. The second set of sufficient conditions deals with the cases where the above sufficient condition fails due to the presence of the choke price. They are two parametric families within H.S.A., which we call "generalized translog" and "constant pass-through." These two parametric families feature the choke price and yet contain CES as the limit case, which allows us to check the robustness of the results under CES.

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[5]Marshall's second law of demand is in general neither sufficient nor necessary for the procompetitive effect.
[6]See, e.g., Feenstra (2003) and Feenstra and Weinstein (2017).
[7]In contrast, under the two other classes of homothetic demand systems studied by Matsuyama and Ushchev (2020a), HDIA, which contains the Kimball demand system used by Baqaee and Fahri (2020), and HIIA, two aggregators are necessary to capture all the competitive pressures. As a result, the dynamics are characterized not by a piecewise linear map, but by a piecewise smooth map, which cannot be solved analytically.
[8]In addition to its single aggregator property, there is another advantage of H.S.A. demand systems, as pointed out by Kasahara and Sugita (2020). That is, the market share (in revenue) functions are the primitive of H.S.A. hence it can be readily identified with the typical firm-level data, which contain revenue, but not the output.

Interestingly, the destabilizing effects become *amplified* as the demand system approaches to CES. And the qualitative properties of the dynamics change *discontinuously* with an arbitrarily small departure from CES. This suggests that, even if an empirically estimated pass-through rate is close to one, using CES as an approximation could be misleading. We also discuss some cross-sectional implications in a multi-market extension. For example, because innovation/entry activities fluctuate more in larger markets, they are not always higher in larger markets than smaller markets. Furthermore, the sale of each product, conditional on surviving idiosyncratic obsolescence shocks, is more volatile in larger markets.

The rest of the paper is organized as follows. In Section 2, we revisit the Judd model under CES, derive a skewed-V map, which generates the equilibrium trajectory, and offer a full characterization of the properties. In doing so, we highlight its key features and explain the intuition why it generates endogenous fluctuations in innovation, why an increase in the (exogenous) constant elasticity of substitution between products has a destabilizing effect, and yet why it is independent of the market size/innovation cost ratio. In Section 3, we formally introduce symmetric H.S.A. demand systems with gross substitutes defined over a continuum of products. Then, we derive the dynamical system for the Judd model under H.S.A., which still features a skewed-V map. In Section 4, we introduce another assumption on H.S.A., Marshall's (weak and strong) 2^{nd} law of demand, which generates the procompetitive effect under H.S.A. In Section 5, we present two propositions. Proposition 1 states that, under H.S.A. with the procompetitive effect, the delayed impact of innovation, $\theta,$ can take the same range of values as under CES, even though it now depends on the market size/innovation cost ratio. Proposition 2 and its Corollary state that the delayed impact of innovation, $\theta,$ is strictly increasing in the market size/innovation cost ratio if the procompetitive effect is combined with the "not too convex" condition. In Section 6, we present two parametric families within H.S.A, "generalized translog" and "constant pass-through," both of which feature the procompetitive effect, the choke price, and contain CES as a limit case. In both families, an explicit calculation allows us to show that an increase in the market size/innovation cost ratio increases the delayed impact of innovation, $\theta,$ and hence has the destabilizing effects on the dynamics of innovation. Interestingly, the destabilizing effects become *amplified* as the demand system approaches to the CES limit within each family. In Section 7, we discuss some cross-sectional implications in a multi-market extension. We conclude in Section 8. Appendices A through E offer some relatively more technical materials.

2. Innovation cycles under CES: Revisiting Judd (1985)

In his seminal work, Judd (1985) developed dynamic extensions of the Dixit-Stiglitz monopolistic competitive model, in which innovators pay a one-time fixed cost of innovation to introduce a new (horizontally differentiated) product, but they hold onto their monopoly power over their own products only for a limited time. Thus, each product is sold initially at the monopoly price, and later at the competitive price. This creates a temporal clustering of innovation activities. Because of free entry to innovation activities, any potential innovator needs to enter when the market for its product is large enough to recover the cost of innovation. The size of the market depends in part on how the products with which it competes are priced. If this innovator chooses to enter when others do, some of its competing products are monopolistically priced. If this innovator enters after others have innovated, on the other hand, the market for its product would be too small to recover the cost of innovation, because competing products are more competitively priced as their innovators lose their monopoly power; the market is too saturated with competitively priced products. So, this innovator would rather enter the market when others do, so that he enjoys his temporary monopoly power while they still hold monopoly, instead of waiting and entering the market after they have lost their monopoly. Or to put it differently, the full impact of aggregate innovations on the competitive pressures occurs with a delay, which each innovator wants to avoid. This creates strategic complementarity in the *timing* of innovation, creating a synchronization of innovation activities and aggregate fluctuations.[9]

[9] It is important to stress that, in the Judd model, the monopoly profit, an incentive to innovate, would be lower if more innovations would take place in the same period, so that contemporaneous innovations are discouraging to innovators. This means that *no* strategic complementarity exists in innovation. What causes a temporal clustering in the Judd model is that, even though contemporaneous innovations are discouraging, they are less so than past innovations, which creates strategic complementarity in *timing* of innovation. In contrast, in the implementation cycle model of Shleifer (1986), temporal clustering occurs due to expectational indeterminacy. This is because an incentive to innovate would be higher if more innovations take place in the same period, so that contemporaneous innovations are encouraging to innovators; such strategic complementarity in innovation creates multiple equilibria, which generates a coordination problem among innovators; if everyone anticipates that everyone else would wait until the next period to innovate, they all wait; if everyone anticipates that everyone else would innovate immediately, they all innovate immediately. Shleifer (1986, footnote 3) explained these differences succinctly, as "Judd's mechanism is almost opposite of mine: innovations in his model repel rather than attract other innovations." The Judd model also differs from the creative destruction model of Aghion-Howitt (1992), which also generates cycles as shown by Benhabib (2004), because innovators are discouraged by future rather than past innovations in their model.

Judd (1985) developed two models to capture this idea, of which we use the one, sketched by Judd (1985, section 4), and later examined in some detail by Deneckere and Judd (1992), for the analytical tractability. What makes this model particularly tractable is the additional assumptions that time is discrete, and that innovators can enjoy their monopoly for only one period, the same period in which they pay the innovation cost. Under this assumption, the equilibrium path can be described by a one-dimensional dynamical system. This is because the state of the economy in each period is summarized by a single variable, how many products the economy has inherited from past innovations, which determines how saturated the market is, and with no possibility of earning future profits, the entry decision of innovators is reduced to a static problem.[10] In this section, we will revisit this version of the Judd model, highlighting the key features of the model, offering a full characterization, and explaining the intuition.

2.1. Representative Household: Time is discrete and denoted by $t \in \{0, 1,2, \dots\}$. The representative household of the economy supplies L units of labor, the only primary factor of production and taken as the *numeraire*, and consumes the single consumption good, C_t, each period. The household has a well-defined intertemporal utility function, $U(C_0, C_1, C_2, \dots)$, but we could leave it unspecified. This is because, in the Judd model as well as in our extension, there exists no aggregate means to save[11]. Hence, the interest rate adjusts endogenously in such a way the representative household spends its income each period, $P_t C_t = L$.

2.2. Production of the Final (Consumption) Good: The competitive industry produces the single consumption good by assembling a continuum of differentiated intermediate inputs, using the CRS technology,

$$C_t = Y_t = F(\mathbf{x}_t),$$

[10]In the other model presented in Judd (1985; sec.3), time is continuous, and monopoly lasts for $0 < T < \infty$, and its equilibrium conditions are described by a system of *delayed differential equations*. In other words, the equilibrium path is described by a dynamical system with an infinite dimensional state space. Although not analytically tractable, Judd showed that there exists a minimum length of the monopoly power, $T_c > 0$, such that for $T_c < T < \infty$, the dynamics of innovation exhibit persistent fluctuations along the equilibrium trajectory for almost all initial conditions. Thus, the discrete time assumption in Judd (1985, section 4) as well as in our extension is not crucial for generating fluctuations.

[11]In this model, there is no asset other than the ownership of the innovating firms, whose market value is equal to zero, because the innovators have to pay the fixed cost of innovation in the same period they earn the monopoly profit, and there is free entry to innovation activities. Introducing other assets into this model, such as physical capital, as in Matsuyama (1999, 2001), or allowing for the innovators to retain the monopoly power more than one period, as in Judd (1985, section 3), would substantially complicate the analysis without adding much insight on the question addressed in this paper.

where $\mathbf{x}_t = \{x_t(\omega);\ \omega \in \Omega_t\}$ is a vector of the intermediate inputs, with Ω_t being the (endogenously determined) set of input varieties available for use in t, and $F(\mathbf{x}_t)$ is strictly increasing, strictly quasi-concave in the interior, and linear homogeneous in $\mathbf{x}_t$ for a given Ω_t. Its unit cost function is

$$P_t = P(\mathbf{p}_t) \equiv \min_{\{x_t(\omega);\ \omega\in\Omega_t\}} \left\{\int_{\Omega_t} p_t(\omega)x_t(\omega)d\omega \,\middle|\, F(\mathbf{x}_t) \geq 1\right\},$$

where $\mathbf{p}_t = \{p_t(\omega);\ \omega \in \Omega_t\}$ is a vector of the input prices, and $P(\mathbf{p}_t)$ is strictly increasing, quasi-concave, and linear homogeneous in $\mathbf{p}_t$ for a given Ω_t. From the unit cost function, one could also recover the CRS production function, as follows:

$$F(\mathbf{x}_t) \equiv \min_{\{p_t(\omega);\ \omega\in\Omega_t\}} \left\{\int_{\Omega_t} p_t(\omega)x_t(\omega)d\omega \,\middle|\, P(\mathbf{p}_t) \geq 1\right\}.$$

Thus either $F(\mathbf{x}_t)$ or $P(\mathbf{p}_t)$ can be used as a primitive of the CRS technology, as long as they satisfy the linear homogeneity, monotonicity, and strict quasi-concavity.

From the Shephard's Lemma, the demand curve for each input can be written as:

$$x_t(\omega) = \frac{\partial P(\mathbf{p}_t)}{\partial p_t(\omega)} Y_t, \tag{1}$$

which can be rewritten to show that the market share of each input is equal to the elasticity of $P(\mathbf{p}_t)$ with respect to its price.

$$\frac{p_t(\omega)x_t(\omega)}{P_t Y_t} = \frac{p_t(\omega)}{P(\mathbf{p}_t)} \frac{\partial P(\mathbf{p}_t)}{\partial p_t(\omega)}. \tag{2}$$

Judd (1985) considers the case where this CRS technology is symmetric CES, following Dixit and Stiglitz (1977; section I), as follows:

$$C_t = Y_t = F(\mathbf{x}_t) = Z\left[\int_{\Omega_t} [x_t(\omega)]^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \tag{3}$$

with $\sigma > 1$, the (constant) elasticity of substitution, and $Z > 0$ a productivity parameter. The corresponding unit cost function is:

$$P_t = P(\mathbf{p}_t) = \frac{1}{Z}\left[\int_{\Omega_t} [p_t(\omega)]^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}}. \tag{4}$$

Hence, eq.(1), the demand curve for each input, becomes

$$x_t(\omega) = \frac{1}{Z}\left[\frac{p_t(\omega)}{ZP(\mathbf{p}_t)}\right]^{-\sigma} Y_t = \frac{[p_t(\omega)]^{-\sigma} L}{[ZP(\mathbf{p}_t)]^{1-\sigma}} = \frac{[p_t(\omega)]^{-\sigma} L}{\int_{\Omega_t} [p_t(\omega)]^{1-\sigma} d\omega} \tag{5}$$

so that the price elasticity of demand for each input is exogenously constant and equal to $\sigma > 1$. And eq.(2), the market share of each input, becomes:

$$\frac{p_t(\omega)x_t(\omega)}{P_tY_t} = \frac{p_t(\omega)}{P(\mathbf{p}_t)}\frac{\partial P(\mathbf{p}_t)}{\partial p_t(\omega)} = \left[\frac{p_t(\omega)}{ZP(\mathbf{p}_t)}\right]^{1-\sigma}.$$

2.3. Differentiated Input Varieties: The set of differentiated inputs available for use in t, Ω_t, changes over time due to *innovation, diffusion,* and *obsolescence*. More specifically, Ω_t is partitioned into Ω_t^m and Ω_t^c. The former, Ω_t^m, is the set of the new inputs introduced & sold exclusively (and monopolistically) by the innovators. They enjoy the monopoly power *for just one period,* the same period in which the innovators pay the innovation cost. The latter, Ω_t^c, is the set of all inputs that the economy inherited at the beginning of period t. Because all these input varieties were innovated before period t, their innovators have already lost their monopoly power, due to diffusion, and hence they are competitively priced.[12] In addition, all the input varieties in $\Omega_t = \Omega_t^m + \Omega_t^c$ are subject to idiosyncratic obsolescence shocks, and only a fraction $\delta \in (0,1)$ survives and carries over to the next period to be in Ω_{t+1}^c.

2.4. Production and Pricing of Differentiated Inputs: Producing one unit of each variety in Ω_t requires ψ units of labor, the numeraire. Thus, the marginal cost of producing each input is equal to ψ. The unit price of all competitively priced input varieties in Ω_t^c is equal to its marginal cost, ψ. Since they all enter symmetrically in the production, they are produced by the same amount, x_t^c, so that:

$$p_t(\omega) = \psi \equiv p^c;\ \ x_t(\omega) \equiv x_t^c \text{ for } \omega \in \Omega_t^c. \tag{6}$$

In contrast, the unit price of all monopolistically supplied input varieties in Ω_t^m is priced at the same exogenously constant markup, as $p_t(\omega) = M\psi$, where $M \equiv \sigma/(\sigma-1)$, because each innovator/monopolist faces the demand curve eq.(5) with the constant price elasticity, $\sigma > 1$. Again, due to the symmetry, they are all produced by the same amount, x_t^m, so that:

$$p_t(\omega) = \frac{\sigma\psi}{\sigma-1} \equiv p^m;\ \ x_t(\omega) \equiv x_t^m \quad \text{for } \omega \in \Omega_t^m. \tag{7}$$

From eqs.(6)-(7),

[12]Even after they have lost the monopoly power, the innovators could remain the sole producers of their innovations. Indeed, this needs to be the case, if we assume that the competitive fringes are required to pay a small fixed cost to produce, and let this fixed cost go to zero; in the limit, the innovators cannot set the price above the marginal cost, due to the *mere* presence of the competitive fringes.

$$\frac{p^c}{p^m} = 1 - \frac{1}{\sigma} < 1; \ \frac{x_t^c}{x_t^m} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma} > 1; \tag{8}$$

and hence the market share of a competitive variety relative to that of a monopolistic variety is

$$\frac{p^c x_t^c}{p^m x_t^m} = \left(1 - \frac{1}{\sigma}\right)^{1-\sigma} \equiv \theta \in (1, e), \tag{9}$$

which is a constant number, θ. It is monotonically increasing in σ, with $\theta \to 1$, as $\sigma \to 1$, and $\theta \to e = 2.718 \ldots$, as $\sigma \to \infty$. It should also be pointed out that θ, though monotonically increasing in σ, changes little in response to σ ($\theta \approx 2.370$ for $\sigma = 4$ and $\theta \approx 2.627$ for $\sigma = 14$).

This constant number, θ, plays a crucial role in the analysis. To understand what it represents, plug the common prices given in eqs.(6)-(7), $p_t(\omega) \equiv p^c$ for $\omega \in \Omega_t^c$ and $p_t(\omega) \equiv p^m$ for $\omega \in \Omega_t^m$, into eq.(4) to obtain the expression for the TFP:

$$\frac{Y_t}{L} = \frac{1}{P_t} = Z[V_t^c (p^c)^{1-\sigma} + V_t^m (p^m)^{1-\sigma}]^{\frac{1}{\sigma-1}} = \frac{Z}{\psi} (V_t)^{\frac{1}{\sigma-1}} \tag{10}$$

where V_t^c and V_t^m denote the measures of Ω_t^c and Ω_t^m, respectively, and

$$V_t \equiv V_t^c + \frac{V_t^m}{\theta}. \tag{11}$$

Here, V_t can be viewed as the "competitive equivalent" mass of input varieties, since Eqs.(10)-(11) show that one competitive variety has the same impact on productivity with those of $\theta > 1$ monopolistic varieties. Thus, the effect of innovation on TFP is initially muted, when the newly introduced inputs are sold at the monopoly price; it reaches its full potential only after their innovators lost their monopoly power, and their innovations become competitively priced. Thus, $\theta - 1 > 0$ measures the *delayed impact of innovation*. This also means that past innovations are more discouraging than contemporaneous innovations to each innovator. To see this, plug the common prices given in eqs.(6)-(7), $p_t(\omega) = p^c$ for $\omega \in \Omega_t^c$ and $p_t(\omega) = p^m$ for $\omega \in \Omega_t^m$ in eq.(5), to obtain the demand curve faced by each innovator in equilibrium:

$$x_t(\omega) = \frac{L\big(p_t(\omega)\big)^{-\sigma}}{V_t^c (p^c)^{1-\sigma} + V_t^m (p^m)^{1-\sigma}} = \frac{L\big(p_t(\omega)\big)^{-\sigma}}{V_t (\psi)^{1-\sigma}},$$

which is inversely related to V_t. Thus, from the point of view of the innovator/monopolist, competing against one competitive variety is equivalent to competing against $\theta > 1$ monopolistic varieties. In other words, θ represents the toughness of competing against a competitive variety, relative to competing against a monopolistic variety. This creates an incentive for innovations to synchronize. Each innovator prefers enjoying its temporary monopoly power, while other innovators are enjoying their temporary monopoly power, i.e., before their innovations become competitively priced. Thus, θ also measures *the force for temporal clustering of innovations*.

2.5. Introduction of New Varieties (Innovation): There is free entry to innovation activities. Anyone can introduce new input varieties at the beginning of each period, which requires F units of labor per variety. Innovations in period t must be rewarded by the monopoly profit earned in period t, as the monopoly power lasts only one period. Thus, unless the gross profit $(p^m - \psi)x_t^m = p^m x_t^m/\sigma = \psi x_t^c/\theta\sigma$ covers the cost of innovation, F, there is no entry/innovation. On the other hand, if there is active entry/innovation, the profit net of the innovation cost must be equal to zero. This can be written as the complementary slackness condition:

$$V_t^m \geq 0; \tag{12}$$
$$F \geq (p^m - \psi)x_t^m = p^m x_t^m/\sigma = \psi x_t^c/\theta\sigma\,;$$
$$V_t^m[(p^m - \psi)x_t^m - F] = V_t^m[p^m x_t^m - (\psi x_t^m + F)] = 0.$$

2.6. Resource Constraint: Labor, the only primary factor of production, is used in the production of intermediate inputs as well as the innovation activities. Thus, the resource constraint of the economy is given by:

$$L = V_t^c(\psi x_t^c) + V_t^m(\psi x_t^m + F).$$

Using eq.(9), eq.(11) and eq.(12), this can be further written as:

$$L = V_t^c(p^c x_t^c) + V_t^m(p^m x_t^m) = V_t(\psi x_t^c)$$

For $V_t^m > 0$, eq.(12) implies $\psi x_t^c = \sigma\theta F$, so that this resource constraint implies $V_t = L/(\sigma\theta F)$. For $V_t^m = 0$, $V_t = V_t^c$. Hence, $V_t \equiv V_t^c + V_t^m/\theta = max\{L/(\sigma\theta F)\,, V_t^c\}$, from which

$$V_t^m = max\left\{\frac{L}{\sigma F} - \theta V_t^c, 0\right\}. \tag{13}$$

Eq.(13) shows that innovations are inactive ($V_t^m = 0$), when $V_t^c \geq L/(\sigma\theta F)$ and active ($V_t^m > 0$) when $V_t^c < L/(\sigma\theta F)$. Thus, $L/(\sigma\theta F)$ can be interpreted as the saturation level of competitive varieties, which kills any incentive to innovate. Eq.(13) also shows that, when innovations are active, $V_t = L/(\sigma\theta F)$ is constant, and hence one additional competitive variety crowds out $\theta > 1$ innovations. Note also that the scale of production of each of competitive and monopolistic varieties, $x_t^c = \sigma\theta(F/\psi)$ and $x_t^m = (\sigma-1)(F/\psi)$, are independent of L. The size of the economy affects only how much innovation takes place.

2.7. Dynamical System: We are now ready to derive the law of motion for the economy. Recall that the economy inherits V_t^c of competitive varieties at the beginning of period t. From eq.(13), this determines innovations, V_t^m. Due to idiosyncratic obsolescence shocks, only a fraction $\delta \in (0,1)$ of all the input varieties produced in period t survive to period $t+1$. Thus,

$$V_{t+1}^c = \delta(V_t^c + V_t^m) = \delta max\left\{\frac{L}{\sigma F} + (1-\theta)V_t^c, V_t^c\right\}.$$

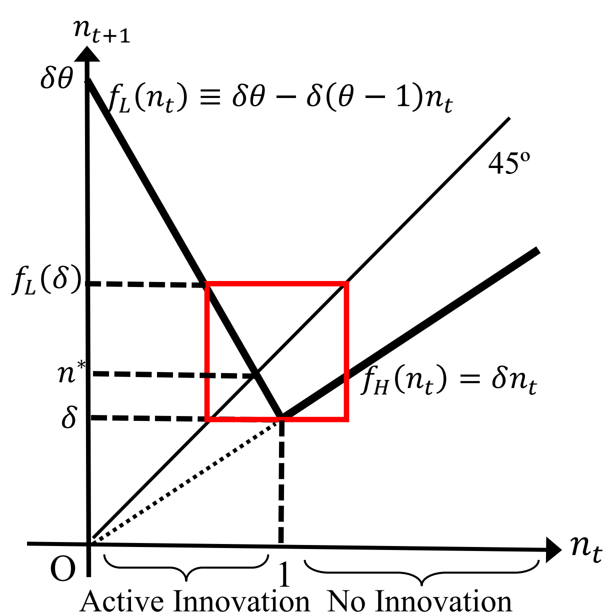
This defines the law of motion for V_t^c. However, it is more convenient to normalize V_t^c with $L/(\sigma\theta F)$, the saturation level of competitive varieties, by defining $n_t \equiv \left(\frac{\sigma\theta F}{L}\right)V_t^c$, which is equal to the market share of competitive varieties, if $n_t \leq 1$, and may be called *the market saturation rate*. Then, the above law of motion is simplified to:

$$n_{t+1} = f(n_t) \equiv \begin{cases} f_L(n_t) \equiv & \delta\theta - \delta(\theta-1)n_t & \text{for } n_t < 1 \\ f_H(n_t) \equiv & \delta n_t & \text{for } n_t > 1 \end{cases} \tag{14}$$

where the two parameters satisfy $(\delta,\theta) \in (0,1)\times(1,e)$. For any initial condition, n_0, the entire equilibrium trajectory of the economy can be obtained by iterating eq.(14).

Figure 1 illustrates eq.(14) for the case of $\delta(\theta-1) > 1$. This dynamical system, eq.(14) is defined by a skewed V-shaped map, a 1-dimensional piecewise linear map $f(n_t)$ with two branches, one decreasing $f_L(n_t)$ and one increasing $f_H(n_t)$. It has two parameters, $\delta \in (0,1)$, the survival rate of each input varieties, and $\theta \in (1,e)$, the market share multiplier due to the loss of monopoly power by its innovator, which also captures the delayed impact of innovations and the force of temporal clustering of innovations.

Figure 1:
The skewed V-map,
drawn for $\delta(\theta-1) > 1$.



The economic intuition behind this map is easy to grasp. Recall that $n_t \equiv (\sigma\theta F/L)V_t^c$ is the market saturation rate, that is, the range of competitive varieties the economy inherited, V_t^c, normalized by the saturation level, $L/(\sigma\theta F)$. Thus, for $n_t > 1$, no innovation takes place. In this phase, n_t shrinks by the factor $\delta < 1$, due to the obsolescence shocks, which is why the map is linear and below the 45º line. Hence, the economy eventually enters the phase, $n_t < 1$, where some innovations take place. In this phase, because an increase in n_t crowds out innovations at the rate equal to $\theta > 1$, the total range of (both competitive *and* monopolistic) input varieties produced is decreasing in n_t by the factor of $\theta - 1$. And because only δ fraction of them survives to the next period, a higher n_t reduces n_{t+1} at the rate equal to $\delta(\theta - 1) > 0$, which is why the map is downward-sloping in this range. (In Figure 1, $\delta(\theta - 1) > 1$.)

2.8. Properties of the Skewed-V map: One major advantage of eq.(14), a skewed V-map, is that its properties can be fully characterized in terms of the two parameters, (δ, θ).[13]

As seen in Figure 1, eq.(14) has a unique **steady state,**

$$n^* \equiv \frac{\delta\theta}{1 + \delta(\theta - 1)} < 1.$$

Its stability depends on $\delta(\theta - 1) > 0$, the slope of $f_L(n_t)$, which is determined by the extent to which innovations in one period discourage those in the next period, which is equal to $\theta - 1$ (the delayed impact of an innovation) multiplied by δ (the probability with which innovated products survive for one period).

For $\delta(\theta - 1) < 1$, this effect dissipates over time, making the unique steady state n^* stable. Indeed, one could easily show that it is not only locally stable but also *globally attracting*; that is, for any initial condition, the equilibrium trajectory converges to n^*. The speed of convergence to the steady state is inversely related to $\delta(\theta - 1)$ and approaches to zero, as $\delta(\theta - 1) \to 1$.

For $\delta(\theta - 1) > 1$, as drawn in Figure 1, innovations in one period discourage more innovations in the next period, making the unique steady state n^* unstable. In this case, starting from any initial condition, the trajectory will eventually enter the interval, $[\delta, f_L(\delta)]$, depicted by the red square in Figure 1, and once entered, it never leaves. In other words, this interval is both

[13] Sushko and Gardini (2010, section 3.1) offers a complete analysis of continuous, piecewise linear maps with two branches, increasing in the lower branch, and decreasing in the upper branch, which they call “skew-tent maps”. By defining $y_t = -n_t$, our skewed-V map can be transformed into a skew-tent map, $y_{t+1} = T(y_t)$.

absorbing and *trapping*. We shall call it *the trapping interval*. Furthermore, for almost all initial conditions, the trajectory exhibits persistent fluctuations within the trapping interval. One could also show that, within this trapping interval, there exists a unique **period-2 cycle,** $n_L^* \leftrightarrow n_H^*$, along which the trajectory oscillates between the phase of active innovation ($n_t = n_L^* < 1$) and the phase of no innovation ($n_t = n_H^* > 1$), where

$$\delta < n_L^* \equiv \frac{\delta^2\theta}{1+\delta^2(\theta-1)} < n^* \equiv \frac{\delta\theta}{1+\delta(\theta-1)} < 1 < n_H^* \equiv \frac{\delta\theta}{1+\delta^2(\theta-1)} < f_L(\delta).$$

The stability of the period-2 cycle, $n_L^* \leftrightarrow n_H^*$, depends on $\delta^2(\theta-1)$, which measures the extent to which innovations in one period discourage those in two periods later, which is equal to $\theta - 1$ (the delayed impact of an innovation) multiplied by δ^2 (the probability with which innovated products survive for two periods).

If $\delta^2(\theta-1) < 1 < \delta(\theta-1)$, the period-2 cycle is stable. Even though $\delta(\theta-1) > 1$ implies that enough of innovations in t survives for one period to discourage innovations in $t+1$, $\delta^2(\theta-1) < 1$ implies that not enough of them survives for two periods to discourage innovations in $t+2$, which makes the period-2 cycle stable. One could also show that the equilibrium trajectory converges to the period-2 cycle for almost all initial conditions. Again, the speed of convergence to the period-2 cycle is inversely related to $\delta^2(\theta-1)$ and approaches to zero, as $\delta^2(\theta-1) \to 1$.

For $\delta^2(\theta-1) > 1$, the period-2 cycle, $n_L^* \leftrightarrow n_H^*$, is unstable. In this case, Deneckere and Judd (1992) pointed out the existence of ergodic chaos in the sense of Lasota and Yorke (1973). In fact, recent advances in piece-wise linear dynamical systems, reviewed by Sushko and Gardini (2010), allow us to say more. That is, there exists a unique **chaotic attractor**,[14] consisting of 2^mcyclic intervals[15], where m is a non-negative integer, which depends on the parameter values. And the trajectory converges to it for almost all initial conditions.

[14]As pointed out in Matsuyama, Sushko, and Gardini (2016, p.529), most existing examples of chaos in economics are *not* attractors, as they rely on Li-Yorke (1975)'s "period-3 implies chaos," which asserts only the existence of a chaotic (i.e., persistently aperiodically fluctuating) trajectory for *some* initial conditions. And the set of the initial conditions leading to such trajectories can be measure zero. In other words, even with the existence of a period-3 cycle, the trajectory may converge to a stable periodic cycle for almost all initial conditions. Here, the trajectory converges to a chaotic attractor for almost all initial conditions.

[15]Along the chaotic attractor consisting of k cyclic intervals, the trajectory visits each interval every k periods but never return to the same value so that it ends up filling each interval.

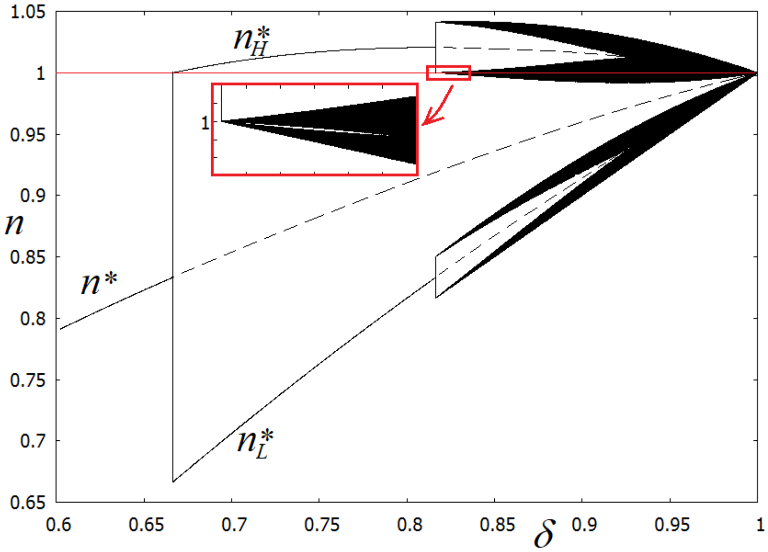


Figure 2:
Effects of an increase in δ (for $\theta = 2.5$). (In courtesy of L.Gardini & I.Sushko)

Figure 2 illustrates these properties of eq.(14) by showing how the unique attractor changes as δ goes up, for $\theta = 2.5$, which corresponds to $\sigma \approx 6.3159$. For $\delta < (\theta - 1)^{-1} = 2/3$, the unique steady state $n^*$is not only stable but also globally attracting. As δ passes $(\theta - 1)^{-1} = 2/3$, n^* becomes unstable, as indicated by the solid graph of n^* switching to a dotted graph. This gives rise to the stable period-2 cycle, $n_L^* \leftrightarrow n_H^*$. At $(\theta - 1)^{-0.5} \approx 0.8165$, the period-2 cycle becomes unstable, as indicated by a pair of the solid graphs of $n_L^* \leftrightarrow n_H^*$ switching to a pair of the dotted graphs. This gives rise to the chaotic attractor, which first consists of 8 cyclic intervals (as indicated in enlargement in the red box), which in turn merge to become the chaotic attractor consisting of 4 cyclic intervals, which in turn merge to become the chaotic attractor consisting of 2 cyclic intervals. Notice that, for $(\theta - 1)^{-0.5} \approx 0.8165 < \delta < 1$, the chaotic attractor always exists. Thus, the chaotic attractor here is robust; there exists no "window of periodicity," unlike in a chaotic system generated by smooth (i.e., C^∞) maps.[16]

[16]As discussed by Matsuyama, Sushko, and Gardini (2016, p.529), chaotic attractors generated by smooth maps are *not* robust. In a smooth dynamical system, the set of parameter values for which a chaotic attractor exists is totally disconnected (i.e., containing no open sets). The chaotic attractor here is robust (i.e., it exists for an open set in the parameter space), since eq.(14) is nonsmooth due to its regime-switching feature. Note also that, in eq.(14), the loss of the stability of the period-2 cycle *immediately* gives rise to the chaotic attractor, without "the period-doubling route to chaos," another familiar feature of smooth dynamical systems.

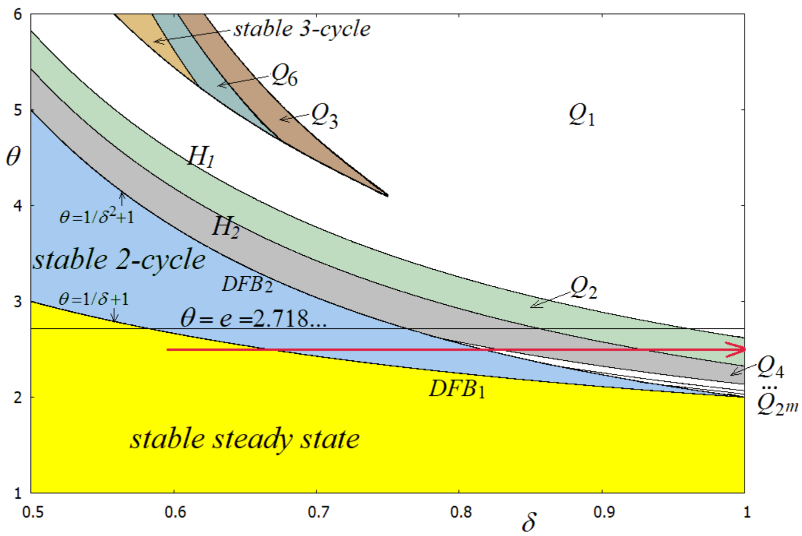


Figure 3: the (*σ*, *θ*)-plane (In courtesy of L.Gardini & I.Sushko)

Q_k indicates the parameter region for the chaotic attractor consisting of $k =$ $(k = 0,1,2\ldots)$ cyclic intervals; Q_{2^m} $(m = 0,1,2\ldots)$ are accumulating to the point $(\delta, \theta) = (1,2)$. (H_1, H_2, DFB_1 and DFB_2 indicate different types of bifurcation occurring at the boundary that separate the regions of different types of the unique attractors.)

In the Judd model, $\theta < e = 2.718$ .... The red arrow, along which $\theta = 2.5$, indicates the experiment illustrated in Figure 2.

Figure 3 illustrates the existence regions of the different types of the unique attractor in the space of the two parameters, (δ, θ). If θ had no upper bound, the skewed-V shape map, eq.(14) could have, as its unique attractor, a stable k-cycle, or a chaotic attractor consisting of k cyclic intervals, where k can be any positive number. However, in the Judd (1985, section 4) model, which assumes CES, $\theta = (1 - \sigma^{-1})^{1-\sigma} < e = 2.718$ ... so that the unique attractor could only be a stable steady state, a stable 2-cycle, or a chaotic attractor of $k = 2^m$ $(m = 0,1,2,\ldots)$ cyclic intervals. Along the red arrow, $\theta = 2.5$, or $\sigma \approx 6.3159$, which corresponds to the thought experiment in Figure 2. Recall that θ, though monotonically increasing in σ, does not change much in response to σ, with $\theta \approx 2.370$ for $\sigma = 4$, and $\theta \approx 2.627$ for $\sigma = 14$. Figure 3 thus indicates that the patterns observed for a wide range of $4 < \sigma < 14$, which roughly corresponds to $2.370 < \theta < 2.627$, is qualitatively similar to those shown in Figure 2.

Figure 3 also shows that, for $\sigma > 2$ (hence $\theta > 2$), a higher δ makes endogenous fluctuations more likely. This is because more of innovations in the current period survives to crowd out innovations in the future. Figure 3 also shows that, for $\delta > (e-1)^{-1} \approx 0.582$, a higher σ (hence a higher θ) makes endogenous fluctuations more likely.[17] This is because the negative impact of competing products becoming competitively priced on the monopoly profit is

[17]For $(e-1)^{-1} \approx 0.582 < \delta < (e-1)^{-1/2} \approx 0.763$, endogenous fluctuations always exhibit a stable period-2 cycle. For $\delta > (e-1)^{-1/2} \approx 0.763$, a period-2 cycle loses the stability and give rise to a chaotic attractor, as σ (hence θ) become higher.

much larger with a higher σ, which gives the innovators stronger incentive to avoid competition by clustering their innovation activities.[18]

Note also that, even inside the region of the stable steady state, both a higher δ and a higher σ (hence a higher θ) slows down the speed of convergence to the steady state, which is inversely related to $\delta(\theta-1)$, making the dynamics more persistent. In other words, it takes longer for the impact of temporal shocks to dissipate, with a higher value of $\delta(\theta-1)$.

2.9. Implications of the CES assumption: One salient feature of the dynamical system, eq.(14), is that it does not depend on L/F, which is a relevant measure of the market size from the point of view of innovators. Even though an increase in L/F increases V_t^m (the number of innovation and the mass of innovators competing against each other), V_t^c (the variety of competitively priced products), and hence the total variety of the inputs produced in the economy, the effects are only proportional.[19] With δ and σ (hence θ) being separate parameters, L/F has no effect on the dynamics of n_t.

This is because, under the CES assumption, eq.(5) or eq.(6), the price elasticity of demand for differentiated inputs, and hence the markup rate, are exogenous, and independent of the market size/innovation cost ratio, in spite of the ample evidence that larger market size and the entry of new firms have the procompetitive effect. Even though the destabilizing impact of a higher θ shown in Figure 3 is suggestive of a potential link between the nature of competition and the patterns of fluctuations in innovation, the CES assumption precludes any possibility that the market size/innovation cost ratio might affect the patterns of fluctuations through its effect on market competition.

3. The Judd Model under H.S.A.

How would the dynamics of innovation change in the presence of the procompetitive effect? To address this question, we now extend the Judd model by using a class of CRS production functions, called H.S.A. As already pointed out in the introduction, this class of CRS

[18]With a higher σ, monopolistic varieties are sold at a lower markup rate, so that the price decline caused by the loss of their monopoly power is smaller. However, with a higher σ, the price decline causes a larger increase in demand. This latter quantity effect dominates the former price effect, which is why θ is increasing in σ, and hence a higher σ has the destabilizing effect.

[19]Simple algebra shows $V_t^c = n_t(L/\sigma\theta F)$; $V_t^m = max\{1-n_t, 0\}(L/\sigma F)$; $V_t = max\{1, n_t\}(L/\sigma\theta F)$, which are all proportional to L/F. From eq.(10), one could also show that TFP and the real wage is monotone increasing in L/F, but the effect is not proportional, unless $\sigma = 2$.

production functions have several advantages. Here, we repeat only one of them. Namely, the Judd model under H.S.A. remains equally tractable as the Judd model under CES, because the dynamical system is still characterized by a skewed-V map. The only difference from the case of CES is that θ, still a constant, now becomes a function of L/F. Thus, by investigating the property of this function, the Judd model under H.S.A. offers a simple way of studying how the market size/innovation cost ratio affects the nature of fluctuations in innovation dynamics through its effect on θ in the presence of the procompetitive effect.

3.1. Symmetric H.S.A. with gross substitutes: In Matsuyama and Ushchev (2017, section 3), we studied a class of homothetic functions that we called *Homothetic with a Single Aggregator* (H.S.A.), and in Matsuyama and Ushchev (2020a, section 3), we restrict this class further by defining over a continuum of varieties and imposing the symmetry and gross substitutability in order to make it applicable to monopolistic competitive settings. More specifically, a symmetric CRS production function, $Y_t = F(\mathbf{x}_t)$, or its unit cost function, $P_t = P(\mathbf{p}_t)$, belongs to the class of H.S.A. if it generates the demand system for inputs such that the market share of each input, which is always equal to the elasticity of $P(\mathbf{p}_t)$ with respect to its own price, as shown in eq.(3), can also be written as

$$\frac{p_t(\omega)x_t(\omega)}{P_tY_t} = \frac{p_t(\omega)}{P(\mathbf{p}_t)}\frac{\partial P(\mathbf{p}_t)}{\partial p_t(\omega)} = s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right). \tag{15}$$

Here, $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$ is the *market share function,* which is twice continuously differentiable[20] and *strictly decreasing* as long as $s(z) > 0$, with $\lim_{z\to\bar{z}} s(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\}$, and $A(\mathbf{p}_t)$ is the *common price aggregator,* linear homogenous in $\mathbf{p}_t$, defined implicitly and uniquely by

$$\int_{\Omega_t} s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right) d\omega = 1, \tag{16}$$

[20]Twice continuous differentiability greatly simplifies the analysis. In Appendix A, we also discuss a piecewise (i.e., kinked) continuously differentiable example to illustrate how the analysis needs to be modified.

which ensures, by construction, that the market shares of all inputs are added up to one.[21] Symmetric CES with gross substitutes is a special case of H.S.A, with $s(z) = \gamma z^{1-\sigma}$ ($\sigma > 1$). Symmetric translog is another special case, with $s(z) = \max\{-\gamma \ln(z/\bar{z}), 0\}$.[22]

Eqs. (15)-(16) state that the market share of an input is decreasing in its *relative price*, which is defined as its own price, $p_t(\omega)$, divided by the *common price aggregator, $A(\mathbf{p_t})$*. Notice that $A(\mathbf{p_t})$ is independent of ω; it is "the average price" against which the relative prices of *all* inputs are measured. In other words, one could keep track of all the cross-price effects in the demand system by looking at a single aggregator, $A(\mathbf{p_t})$, which is the key feature of H.S.A. The assumption that the market share function, $s(\cdot)$, is independent of ω is not a defining feature of H.S.A.; it is due to the symmetry of the underlying production function that generates this demand system. The assumption that it is strictly decreasing in $z < \bar{z}$ means that inputs are gross substitutes. Furthermore, if $\bar{z} < \infty$, $\bar{z}A(\mathbf{p_t})$ is the choke price, at which demand for a variety goes to zero.

The unit cost function, $P(\mathbf{p})$, behind this H.S.A. demand system can be obtained by integrating eq.(15), which yields

$$\ln\left(\frac{P(\mathbf{p})}{A(\mathbf{p})}\right) = \text{const.} - \int_{\Omega} \left[\int_{p(\omega)/A(\mathbf{p})}^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi\right] d\omega.$$

This unit cost function, $P(\mathbf{p})$, satisfies the linear homogeneity, monotonicity, and strict quasi-concavity in the interior, and so does the corresponding production function, $F(\mathbf{x})$, which follows from Matsuyama and Ushchev (2017; Proposition 1-i)). This guarantees the existence of the underlying CRS technology $P(\mathbf{p})$, that generates this H.S.A. demand system. In the case of CES, it is easy to verify that $P(\mathbf{p}) = cA(\mathbf{p})$, where $c > 0$ is a constant. However, it is important

[21]For $A(\mathbf{p_t})$ to be well-defined for all $\mathbf{p_t} = \{p_t(\omega);\ \omega \in \Omega_t\}$, the Lebesgue measure of Ω_t, $\Lambda(\Omega_t)$, must satisfy $\Lambda(\Omega_t)\lim_{z\to 0} s(z) > 1$. If $\lim_{z\to 0} s(z) = \infty$, as in CES and translog, this inequality always holds. To ensure this inequality along the equilibrium path even if $\lim_{z\to 0} s(z) < \infty$, we will assume $L/F > \lim_{z\to 0}\{\zeta(z)/s(z)\}$, where $\zeta(z) \equiv 1 - zs'(z)/s(z)$ is the price elasticity function defined later.

[22]For $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$, satisfying the above conditions, a class of the market share functions, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$, generate the same demand system with the same common price aggregator. We just need to renormalize the indices of varieties, as $\omega' = \gamma\omega$, so that $\int_{\Omega_t} s_\gamma(p_t(\omega)/A(\mathbf{p_t}))d\omega = \int_{\Omega_t} s(p_t(\omega')/A(\mathbf{p_t}))d\omega' = 1$. In this sense, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$, are all equivalent. Note also that a class of the market share functions, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$, generate the same demand system, with $A_\lambda(\mathbf{p_t}) = \lambda A(\mathbf{p_t})$, because $s_\lambda(p_t(\omega)/A_\lambda(\mathbf{p_t})) = s(\lambda\, p_t(\omega)/A_\lambda(\mathbf{p_t})) = s(p_t(\omega)/A(\mathbf{p_t}))$. In this sense, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$ are all equivalent. Using these equivalences, for example, one could obtain the CES case with $s(z) = z^{1-\sigma}$ ($\sigma > 1$) by setting $\gamma = 1$ and the translog case, with $s(z) = \max\{-\ln(z/\bar{z}), 0\}$ by setting $\gamma = 1$ and $\lambda = 1/\bar{z} = 1$, without loss of generality.

to note that, with the sole exception of CES, $P(\mathbf{p}) \neq cA(\mathbf{p})$ for any constant $c > 0$, as shown in Matsuyama and Ushchev (2020a; Corollary 2 of Lemma 2)[23]. This should not come as a total surprise. After all, $A(\mathbf{p})$ is the inverse measure of competitive pressures, which fully captures the *cross-price effects* in the demand system, while $P(\mathbf{p})$ is the inverse measure of TFP, which fully captures the *productivity (or welfare) effects* of price changes; there is no reason to think *a priori* that they should move together.

We are now ready to proceed with the analysis of the Judd model under H.S.A.

3.2. Pricing of Differentiated Varieties*:* All competitive varieties are priced at the marginal cost, $p_t(\omega) = \psi = p^c$ and hence their relative prices are $z_t^c \equiv \psi/A(\mathbf{p}_t)$ for $\omega \in \Omega_t^c$. For monopolistic varieties, $\omega \in \Omega_t^m$, from eq.(15), each monopolist/innovator faces the demand curve,

$$x_t(\omega) = \frac{P_t Y_t}{p_t(\omega)} s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right) = \frac{L}{p_t(\omega)} s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right).$$

Hence it sets the price $p_t(\omega)$ to maximize the profit,

$$(p_t(\omega) - \psi)x_t(\omega) = \left(1 - \frac{\psi}{p_t(\omega)}\right) s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right) L,$$

holding $A(\mathbf{p}_t)$ as given. Or equivalently, it sets its relative price $z_t(\omega) \equiv p_t(\omega)/A(\mathbf{p}_t)$ to solve:

$$max_{z_t(\omega)} \left(1 - \frac{z_t^c}{z_t(\omega)}\right) s\big(z_t(\omega)\big)L \ \equiv max_{z_t(\omega)} \ \pi(z_t(\omega); z_t^c)L \equiv \ \hat{\pi}(z_t^c)L$$

holding $z_t^c \equiv \psi/A(\mathbf{p}_t)$ as given. Here, $\pi(z_t(\omega); z_t^c)$ is the profit per unit of the aggregate expenditure, L, as a function of its relative price, and $\hat{\pi}(z_t^c) \equiv max_{z_t(\omega)} \ \pi(z_t(\omega); z_t^c)$ is the maximized profit per unit of the aggregate expenditure. Thus, the monopoly price needs to satisfy both the following first-order condition (FOC) and second-order condition (SOC):

FOC: $$z_t(\omega)\left[1 - \frac{1}{\zeta\big(z_t(\omega)\big)}\right] = z_t^c$$

SOC: $$\frac{z_t(\omega)\zeta'\big(z_t(\omega)\big)}{\zeta\big(z_t(\omega)\big)} > 1 - \zeta\big(z_t(\omega)\big)$$

where

[23] This holds also for asymmetric H.S.A., as well as H.S.A. with gross complements. See Matsuyama and Ushchev (2017; Proposition 1-iii).

$$\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1 \tag{17}$$

is the price elasticity of the demand curve for a particular variety, which is a function of its relative price; it is inversely related to the markup rate, $M(z) \equiv \zeta(z)/(\zeta(z)-1)$. Note that $\zeta(z) > 1$ is well-defined and continuously differentiable for $z \in (0, \bar{z})$. Conversely, any continuously differentiable $\zeta: (0, \bar{z}) \to (1, \infty)$, satisfying $\lim_{z \to \bar{z}} \zeta(z) = \infty$ if $\bar{z} < \infty$, can be used as a primitive of symmetric H.S.A. production functions with gross substitutes with

$$s(z) = \exp\left[\int_c^z \frac{1-\zeta(\tau)}{\tau} d\tau\right], z \in (0, \bar{z}),$$

where $c \in (0, \bar{z})$ is a constant.

FOC and SOC are sufficient for a local optimum, but generally not for the global optimum. In what follows, we avoid the need to deal with local but not global optima by assuming:[24]

(A1) $$\zeta(z) - 1 + \frac{z\zeta'(z)}{\zeta(z)} > 0, \qquad \text{for } z \in (0, \bar{z}).$$

Lemma: (A1) is equivalent to each of the following three statements:

i) $z\left(1 - \frac{1}{\zeta(z)}\right)$ is strictly increasing in $z \in (0, \bar{z})$.

ii) For any $z^c \in (0, \bar{z})$, $\pi(z; z^c) \equiv \left(1 - \frac{z^c}{z}\right) s(z)$ has a single peak at $z^m \in (z^c, \bar{z})$, given by $z^m\left(1 - \frac{1}{\zeta(z^m)}\right) \equiv z^c$.

iii) $\frac{s(z)}{\zeta(z)}$ is strictly decreasing in $z \in (0, \bar{z})$.

Proof: The equivalence of (A1) and i) follows from $\frac{d\left(z\left(1-\frac{1}{\zeta(z)}\right)\right)}{dz} = \frac{1}{\zeta(z)}\left[\zeta(z) - 1 + \frac{z\zeta'(z)}{\zeta(z)}\right]$. The equivalence of i) and ii) follows from $\frac{d}{dz}\pi(z; z^c) = \frac{s(z)\zeta(z)}{z^2}\left[z^c - z\left(1 - \frac{1}{\zeta(z)}\right)\right]$. Finally, the equivalence of (A1) and iii) follows from $\frac{d\left(\ln \frac{s(z)}{\zeta(z)}\right)}{d \ln z} = \frac{zs'(z)}{s(z)} - \frac{z\zeta'(z)}{\zeta(z)} = 1 - \zeta(z) - \frac{z\zeta'(z)}{\zeta(z)}$. ■

[24]This is mostly for the expositional simplicity. Even if (A1) is violated, much of the analysis would go through. However, the derivation would become far more involved. This is because, for a finite (hence, non-generic) set of the parameter values, different monopolistic varieties are sold at different prices and by different amounts for the same profit. As a result, a change in the parameters could cause discrete jumps in endogenous variables in comparative statics. See Appendix C for an example.

Thus, under (A1), ii) in Lemma holds so that all the innovators sets the same price, $p_t(\omega) = p_t^m$, and hence the same relative price, $z_t(\omega) = z_t^m \equiv p_t^m / A(\mathbf{p}_t)$, given by the FOC,

$$z_t^m \left[1 - \frac{1}{\zeta(z_t^m)}\right] = z_t^c, \tag{18}$$

which automatically satisfies the SOC. Furthermore, from i) in Lemma, z_t^m is continuously differentiable and strictly increasing in z_t^c, and *vice versa.*

3.3. Introduction of New Varieties (Innovation): From eq.(18), the maximized profit is written as,

$$\hat{\pi}(z_t^c)L \equiv \pi(z_t^m; z_t^c)L \equiv \left(1 - \frac{z_t^c}{z_t^m}\right) s(z_t^m)L = \frac{s(z_t^m)}{\zeta(z_t^m)} L.$$

This is because each innovator earns $s(z_t^m)$ fraction of the aggregate expenditure, $P_t Y_t = L$, of which $1 - 1/\zeta(z_t^m)$ fraction is paid to the production cost, and the remaining fraction, $1/\zeta(z_t^m)$, goes to the profit. The free entry (innovation) complementarity slackness condition is thus:

$$V_t^m \geq 0;\ F \geq \frac{s(z_t^m)}{\zeta(z_t^m)} L;\ V_t^m \left[F - \frac{s(z_t^m)}{\zeta(z_t^m)} L\right] = 0,$$

which corresponds to eq.(12) under CES. In what follows, let us assume

$$\frac{F}{L} < \lim_{z \to 0} \frac{s(z)}{\zeta(z)}.$$

which implies, with iii) in Lemma, that $\underline{z}^m > 0$ can be defined uniquely by

$$\frac{s(\underline{z}^m)}{\zeta(\underline{z}^m)} \frac{L}{F} \equiv 1, \tag{19}$$

and it is strictly increasing in L/F. Then, the complementarity condition can be rewritten as:

$$V_t^m \geq 0;\ z_t^m \geq \underline{z}^m;\ V_t^m [z_t^m - \underline{z}^m] = 0.$$

Furthermore, from eq.(18) and i) in Lemma, this also implies

$$z_t^c = z_t^m \left[1 - \frac{1}{\zeta(z_t^m)}\right] \geq \underline{z}^c \equiv \underline{z}^m \left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]. \tag{20}$$

Since the market share of each monopolistic variety is $s(z_t^m)$ and that of each competitive variety is $s(z_t^c)$, the adding up constraint, eq. (16), can now be rewritten as:

$$V_t^m s\left(\frac{p_t^m}{A(\mathbf{p}_t)}\right) + V_t^c s\left(\frac{\psi}{A(\mathbf{p}_t)}\right) = V_t^m s(z_t^m) + V_t^c s(z_t^c) = 1. \tag{21}$$

First, consider the case where $V_t^m > 0 \Longrightarrow z_t^m = \underline{z}^m;\ z_t^c = \underline{z}^c$. Then, eq.(21) becomes

$$V_t^m = \frac{1 - s(\underline{z}^c)V_t^c}{s(\underline{z}^m)} = \theta\left(\frac{1}{s(\underline{z}^c)} - V_t^c\right) > 0$$

where $\theta \equiv s(\underline{z}^c)/s(\underline{z}^m) > 1$ is the market share of a competitive variety, relative to a monopolistic variety. Next, consider the case where $V_t^m = 0$. Then, eq.(21) becomes $V_t^c s(z_t^c) = 1$. Because $z_t^c \geq \underline{z}^c$, this implies

$$V_t^m = 0 \iff V_t^c = \frac{1}{s(z_t^c)} \geq \frac{1}{s(\underline{z}^c)}.$$

By putting these two cases together, we have

$$V_t^m = max\left\{\theta\left(\frac{1}{s(\underline{z}^c)} - V_t^c\right), 0\right\} = max\left\{\frac{L}{\zeta(\underline{z}^m)F} - \theta V_t^c, 0\right\}, \tag{22}$$

which corresponds to eq.(13) under CES. Thus, $1/s(\underline{z}^c)$ can be viewed as the saturation level of competitive varieties, which kills any incentive to innovate.

3.4. Dynamical System: By following the same step as in the case of CES, from eq.(22), we obtain the law of motion for V_t^c,

$$V_{t+1}^c = \delta(V_t^c + V_t^m) = \delta max\left\{\frac{\theta}{s(\underline{z}^c)} + (1-\theta)V_t^c, V_t^c\right\}, \tag{23}$$

where $\delta \in (0,1)$ is the survival rate of each variety. Now, divide the measure of competitive varieties, V_t^c, by the saturation level, $1/s(\underline{z}^c)$, to define the market saturation rate, $n_t \equiv s(\underline{z}^c)V_t^c$, which is also equal to the market share of all competitive varieties for $n_t \leq 1$. The above law of motion is then rewritten as the dynamical system in n_t as follows:

$$n_{t+1} = f(n_t) \equiv \begin{cases} f_L(n_t) \equiv & \delta(\theta + (1-\theta)n_t) \quad \text{for } n_t \leq 1 \\ f_H(n_t) \equiv & \delta n_t \quad \text{for } n_t \geq 1 \end{cases} \tag{24}$$

where we recall $\theta \equiv s(\underline{z}^c)/s(\underline{z}^m) > 1$ is the market share of a competitive variety relative to a monopolistic variety, which measures the *delayed impact of innovations* and the *force of temporal clustering of innovations*.

Notice that eq.(14) and eq.(24) are identical, both characterized by the skewed-V map, with the same two parameters, (δ, θ). However, there is one crucial difference. Under H.S.A., $\theta \equiv s(\underline{z}^c)/s(\underline{z}^m) > 1$ is a function of L/F, $\Theta(L/F)$, which can be defined implicitly as:

$$\Theta\left(\frac{L}{F}\right) \equiv \frac{s\left(\underline{z}^m\left[1-\frac{1}{\zeta(\underline{z}^m)}\right]\right)}{s(\underline{z}^m)} > 1;\ \frac{s(\underline{z}^m)}{\zeta(\underline{z}^m)}\frac{L}{F} \equiv 1, \tag{25}$$

using eq.(19) and eq.(20). From iii) in Lemma, eq.(18) shows that $\underline{z}^m$ is increasing in L/F. Thus, a larger market size/innovation cost ratio allows the innovators to break even with a higher relative price. Under CES, $\zeta(\underline{z}^m) = \sigma$ and $s(z) = (\lambda z)^{1-\sigma}$, so that $\theta = (1-\sigma^{-1})^{1-\sigma} \in (1, e)$, which is independent of $\underline{z}^m$. Generally, however, L/F affects $\theta \equiv s(\underline{z}^c)/s(\underline{z}^m) \equiv \Theta(L/F) > 1$ through its effect on $\underline{z}^m$. And through its effect on $\theta \equiv \Theta(L/F) > 1$, L/F affects the nature of fluctuations. Since a change in L/F keeps eq.(14) otherwise intact, it suffices to study the property of this function in order to identify the effect of the market size/innovation cost ratio.

4. Procompetitive Effect under H.S.A.

Before proceeding, we introduce another assumption.

(A2) $\zeta'(z) \geq 0$ for all $z \in (0, \bar{z})$.

That is, for a fixed $A = A(\mathbf{p}_t)$, the price elasticity of demand for each variety may go up but never go down along its demand curve, as its price goes up. The property is often referred to as Marshall's 2nd law of demand. In what follows, we shall call (A2) the **weak (strong) 2nd law**, if the inequality is (A2) holds weakly (strictly). Since $\zeta(z) > 1$, (A2) implies (A1), from which each of the three equivalent statements in Lemma follows.

What are the implications of (A2) on the firms' pricing behavior? First, when innovation is active and hence there are monopolistic varieties, $V_t^m > 0, z_t^m = \underline{z}^m$ and hence they are sold at the price given by:

$$p_t^m\left[1-\frac{1}{\zeta(\underline{z}^m)}\right] = \psi \Leftrightarrow p_t^m = \frac{\zeta(\underline{z}^m)}{\zeta(\underline{z}^m)-1}\psi = M(\underline{z}^m)\psi.$$

Since $\underline{z}^m$ is a monotone increasing function of L/F, the strong 2nd Law implies that monopolistic varieties are sold at a lower markup rate, with a higher L/F, and hence the larger market size has a procompetitive effect. The weak 2nd Law rules out the possibility of the large market size having an anti-competitive effect.

Second, let us temporarily assume that the marginal cost of production depends on varieties, $\psi(\omega)$, so that FOC of monopoly pricing becomes:

$$p_t^m(\omega)\left[1-\frac{1}{\zeta\big(z_t(\omega)\big)}\right]=\psi(\omega)\Leftrightarrow p_t^m(\omega)=M\big(z_t(\omega)\big)\psi(\omega).$$

where $z_t(\omega) = p_t(\omega)/A(\mathbf{p}_t)$. By totally log-differentiating this expression, we obtain:

$$\mathrm{dln}\, p_t^m(\omega)=\frac{1}{1+\Delta}\mathrm{dln}\,\psi(\omega)+\frac{\Delta}{1+\Delta}\mathrm{dln}\,A(\mathbf{p}_t)$$

where

$$\Delta\equiv-\frac{\mathrm{dln}\,M\big(z_t(\omega)\big)}{\mathrm{dln}\,z_t(\omega)}=\frac{z_t(\omega)\zeta'\big(z_t(\omega)\big)}{\big[\zeta\big(z_t(\omega)\big)-1\big]\zeta\big(z_t(\omega)\big)},$$

where (A1) implies $1+\Delta>0$. This shows how the pricing of a monopolistic variety responds to a change in its own marginal cost, $\psi(\omega)$, as well as a change in the pricing of competing varieties, $A(\mathbf{p}_t)$. Under the strong 2nd law, $\Delta>0$, and hence

$$0<\frac{\mathrm{dln}\,p_t^m(\omega)}{\mathrm{dln}\,A(\mathbf{p}_t)}=\frac{\Delta}{1+\Delta}<1.$$

Thus, an increase in $A(\mathbf{p}_t)$ leads to an increase in $p_t^m(\omega)$. In other words, the pricing of monopolistic varieties satisfies strategic complementarity in pricing.[25] Each monopolist responds to an increase in the prices of competing inputs by increasing its price. At the same time, holding the pricing of competing inputs, $A(\mathbf{p}_t)$, fixed, the strong 2nd law ($\Delta>0$) implies that $p_t^m(\omega)$ responds less than proportionately to a change in its marginal cost, $\psi(\omega)$.

$$0<\frac{\mathrm{dln}\,p_t^m(\omega)}{\mathrm{dln}\,\psi\,(\omega)}=\frac{1}{1+\Delta}<1,$$

which implies that, when the marginal cost goes up, the markup rate would have to decline, *unless* the prices of competing varieties would also go up. This implies (firm-level) incomplete (i.e., less than 100%) pass-through.[26] This also means that, in cross-section of firms, more productive firms have higher markup rates. The weak 2nd law rules out the case of strategic substitutes and more than 100% pass-through at the firm level with less productive firms having higher markup rates.

[25]That is, firms respond to an increase in the prices of competing products by raising their prices/markup rates.

[26]That is, in cross-section of firms, marginal costs are negatively correlated with the markup rates. In other words, more productive firms have higher markup rates. Note that this need not imply an incomplete pass-through at the industrial level. If the marginal cost of all firms change equally, $\mathrm{dln}\,\psi\,(\omega)=\mathrm{dln}\,\psi$, the prices of all varieties respond proportionately to a change in ψ, $\mathrm{dln}\,p_t(\omega)=\mathrm{dln}\,\psi=\mathrm{dln}\,A(\mathbf{p}_t)$, in equilibrium. Thus, a uniform increase in the marginal cost has no effect on $z_t(\omega)$, hence on the markup rate, implying a complete pass-through.

5. Two Propositions

We now consider two key questions about the implications of extending the Judd model from CES to H.S.A. and offer two propositions to answer them.

The first question asks whether the Judd model of innovation cycles under H.S.A. could exhibit types of dynamic paths, which cannot be generated under CES. This boils down to the question of whether θ can be greater than $e = 2.718 \ldots$, which is the upper bound of $\theta = (1-\sigma^{-1})^{1-\sigma}$, under CES. Proposition 1 states that the answer is negative under (A2).

Proposition 1. Under (A2), $\theta \in (1, e)$, where $e = 2.718 \ldots$

Proof: Consider the following identity:

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \exp\left[\int_{\underline{z}^m}^{\underline{z}^c} \frac{1-\zeta(\tau)}{\tau} d\tau\right] = \exp\left[\int_{\underline{z}^c}^{\underline{z}^m} \frac{\zeta(\tau)-1}{\tau} d\tau\right].$$

Under (A2), $\zeta(\cdot)$ is non-decreasing, and hence:

$$\exp\left[\int_{\underline{z}^c}^{\underline{z}^m} \frac{\zeta(\tau)-1}{\tau} d\tau\right] \leq \exp\left[\left(\zeta(\underline{z}^m)-1\right)\int_{\underline{z}^c}^{\underline{z}^m} \frac{d\tau}{\tau}\right] = \exp\left[\left(\zeta(\underline{z}^m)-1\right)\ln\left(\frac{\underline{z}^m}{\underline{z}^c}\right)\right]$$

$$= \exp\left[\ln\left(1-\frac{1}{\zeta(\underline{z}^m)}\right)^{1-\zeta(\underline{z}^m)}\right] = \left(1-\frac{1}{\zeta(\underline{z}^m)}\right)^{1-\zeta(\underline{z}^m)}$$

which is strictly increasing in $\zeta(\underline{z}^m)$, and converges to e, as $\zeta(\underline{z}^m) \to \infty$. ■

The intuition behind this result is simple. Under (A2), price elasticities can become only smaller at lower prices. Hence, when a monopolistic variety becomes competitively priced, an increase in the market share caused by a drop in the price could only be smaller compared to the case of CES, not larger. Thus, it has the same upper bound, $\theta < e$.

Without (A2), however, θ can be arbitrarily large: see Appendix B for an example. Thus, the Judd model under H.S.A. in principle could generate stable cycles of any period, or robust chaotic attractors with any positive number of cyclic intervals.[27]

The second question is when a larger market size/innovation cost ratio, L/F, has destabilizing effect on the dynamics of innovation. Since the dynamic behavior becomes more

[27] For example, one could see in Figure 3 the range of θ, which generates a stable cycle of period 3, a robust chaotic attractor with six cyclic intervals, or a robust chaotic attractor of three cyclic intervals.

unstable as θ becomes large, this boils down to the question of when $\Theta(L/F)$ is increasing in L/F.

Proposition 2: If $\zeta(\cdot) - 1$ is monotone and log-concave over an interval containing $(\underline{z}^c, \underline{z}^m)$, $\Theta(L/F)$ is increasing in L/F. If at least one of the monotonicity and the log-concavity conditions is strict, $\Theta(L/F)$ is strictly increasing in L/F.

Proof: Since $\underline{z}^m$ is strictly increasing in L/F, $\theta \equiv \Theta(L/F)$ is strictly increasing in L/F, if and only if $\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \frac{s\left(\underline{z}^m\left[1-\frac{1}{\zeta(\underline{z}^m)}\right]\right)}{s(\underline{z}^m)}$ is strictly increasing in $\underline{z}^m$. By log-differentiating $\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)}$ with respect to $\underline{z}^m$,

$$\frac{d\ln\theta}{d\ln(\underline{z}^m)} = [1-\zeta(\underline{z}^c)]\frac{d\ln(\underline{z}^c)}{d\ln(\underline{z}^m)} - [1-\zeta(\underline{z}^m)]$$

$$= [1-\zeta(\underline{z}^c)]\left\{1 + \frac{d\ln\left[1-\frac{1}{\zeta(\underline{z}^m)}\right]}{d\ln(\underline{z}^m)}\right\} - [1-\zeta(\underline{z}^m)]$$

$$= \zeta(\underline{z}^m) - \zeta(\underline{z}^c) - \frac{\zeta(\underline{z}^c)-1}{\zeta(\underline{z}^m)-1}\frac{\underline{z}^m\zeta'(\underline{z}^m)}{\zeta(\underline{z}^m)}.$$

Since the mean value theorem implies

$$\zeta(\underline{z}^m) - \zeta(\underline{z}^c) = \zeta'(\tilde{z})(\underline{z}^m - \underline{z}^c) \quad \text{for some } \tilde{z} \in (\underline{z}^c, \underline{z}^m),$$

and the monotonicity of $\zeta(\cdot)$ implies

$$\zeta'(\underline{z}^m)[\zeta(\tilde{z}) - \zeta(\underline{z}^c)] \geq 0,$$

the above expression can be further rewritten as:

$$\frac{d\ln\theta}{d\ln(\underline{z}^m)} = \zeta(\underline{z}^m) - \zeta(\underline{z}^c) - \frac{\zeta(\underline{z}^c)-1}{\zeta(\underline{z}^m)-1}\frac{\underline{z}^m\zeta'(\underline{z}^m)}{\zeta(\underline{z}^m)}$$

$$= \zeta'(\tilde{z})(\underline{z}^m - \underline{z}^c) - \frac{\zeta(\underline{z}^c)-1}{\zeta(\underline{z}^m)-1}\frac{\underline{z}^m\zeta'(\underline{z}^m)}{\zeta(\underline{z}^m)}$$

$$= \zeta'(\tilde{z})\frac{\underline{z}^m}{\zeta(\underline{z}^m)} - \frac{\zeta(\underline{z}^c)-1}{\zeta(\underline{z}^m)-1}\frac{\underline{z}^m\zeta'(\underline{z}^m)}{\zeta(\underline{z}^m)}$$

$$= \left[\frac{\zeta'(\tilde{z})}{\zeta'(\underline{z}^m)} - \frac{\zeta(\underline{z}^c)-1}{\zeta(\underline{z}^m)-1}\right]\frac{\underline{z}^m\zeta'(\underline{z}^m)}{\zeta(\underline{z}^m)} \geq \left[\frac{\zeta'(\tilde{z})}{\zeta'(\underline{z}^m)} - \frac{\zeta(\tilde{z})-1}{\zeta(\underline{z}^m)-1}\right]\frac{\underline{z}^m\zeta'(\underline{z}^m)}{\zeta(\underline{z}^m)}$$

$$= \left[\frac{\zeta'(\tilde{z})}{\zeta(\tilde{z}) - 1} - \frac{\zeta'(\underline{z}^m)}{\zeta(\underline{z}^m) - 1}\right] \frac{\underline{z}^m \left(\zeta(\tilde{z}) - 1\right)}{\zeta(\underline{z}^m)} \geq 0,$$

where the term in the square bracket is non-negative because the log-concavity of $\zeta(\cdot) - 1$ means that $\frac{\zeta'(\cdot)}{\zeta(\cdot)-1}$ is decreasing. This proves $\frac{d \ln \theta}{d \ln(\underline{z}^m)} \geq 0$ and hence $\Theta'(L/F) \geq 0$.

Furthermore, if at least one of the monotonicity and the log-concavity conditions holds strictly, one of the two inequalities above holds strictly, from which $\frac{d \ln \theta}{d \ln(\underline{z}^m)} > 0$ and $\Theta'(L/F) > 0$. follows. ■

Corollary: Under the weak (strong) 2nd Law, $\theta \equiv \Theta(L/F)$ is strictly increasing in L/F, if $\zeta(\cdot) - 1$ is strictly (weakly) log-concave.

Note that the log-concavity of $\zeta(\cdot) - 1$ is weaker than the concavity of $\zeta(\cdot) - 1$, and hence the concavity of $\zeta(\cdot)$. For a thrice-continuously differentiable $s(\cdot)$, and hence twice-continuously differentiable $\zeta(\cdot)$, $\zeta(\cdot) - 1$ is weakly log-concave if and only if

$$\zeta''(\cdot) \leq \frac{\left(\zeta'(\cdot)\right)^2}{\zeta(\cdot) - 1},$$

which can be interpreted as $\zeta(\cdot)$ being "not too convex."

What is the intuition behind Corollary? Under (A1), a higher L/F leads to a continuous increase in both $\underline{z}^m$ and $\underline{z}^c$. Under the strong 2nd Law, $\zeta'(\cdot) > 0$, the procompetitive effect leads to an increase in $\zeta(\underline{z}^m)$ as well as an increase in $\zeta(z)$ over the range, $(\underline{z}^c, \underline{z}^m)$. The former implies a lower markup rate, and hence the price drop due to the loss of monopoly is smaller, which contributes to a smaller θ. The latter implies the market share responds more to the price drop, which contributes to a larger θ. As we know from the CES case, if the price elasticity would go up uniformly, the latter quantity effect dominates the former price effect, and θ would go up. If $\zeta(\cdot)$ is not "too convex," $\zeta(\underline{z}^m)$ does not go up too much faster than $\zeta(z)$ over the range, $(\underline{z}^c, \underline{z}^m)$, so that the former price effect does not dominate the latter quantity, and hence, θ becomes increasing in L/F.[28]

[28]The above argument also tells us how things can go "wrong," when the log-concavity of $\zeta(\cdot) - 1$ fails. See Appendix D for a pathological example, where θ is strictly decreasing in L/F in spite of the strong 2nd law. In Appendix E, we show two parametric families, which satisfy the log-concavity condition in addition to the strong 2nd law, and as such, they demonstrate the power of Proposition 2. Both families contain CES as a limit case.

6. Two Parametric Families: Generalized Translog and Constant Pass-Through

The log-concavity of $\zeta(\cdot)-1$ implies $\bar{z}=\infty$, hence rules out the choke price. This section presents two parametric families of H.S.A., which satisfy the strong 2nd Law with the choke price, hence violate the log-concavity condition. Yet, they are tractable enough that θ can be expressed explicitly and shown to be strictly increasing in L/F. Thus, these two parametric families demonstrate that the log-concavity is merely *sufficient, but not necessary* for a larger market size/innovation cost ratio to be destabilizing through the procompetitive effect. We call the first family, "generalized translog," because it contains the translog as a special case. We call the second family, "constant pass-through," because it implies that the elasticity of the monopoly price with response to the marginal cost, the pass-through rate, is constant and less than one. Even though both families feature the choke price, each contains CES as a limit case.[29] Interestingly, the destabilizing effects of the market size/innovation cost ratio through the procompetitive effect become *amplified* as the demand system approaches to the CES limit within each family. In other words, the qualitative properties of the dynamics change *discontinuously* with an arbitrarily small departure from CES.

Example 1: Generalized Translog

$$s(z)=\begin{cases}\gamma\left(1-\dfrac{\sigma-1}{\eta}\ln\left(\dfrac{z}{\beta}\right)\right)^{\eta}=\gamma\left(\dfrac{1-\sigma}{\eta}\ln\left(\dfrac{z}{\bar{z}}\right)\right)^{\eta} & \text{for } z<\bar{z}\equiv\beta e^{\frac{\eta}{\sigma-1}}\\[2ex] 0 & \text{for } z\geq\bar{z}\equiv\beta e^{\frac{\eta}{\sigma-1}}\end{cases}$$

where $\beta>0, \eta>0;\ \sigma>1$. Then,

$$\zeta(z)=1+\frac{\sigma-1}{1-\dfrac{\sigma-1}{\eta}\ln\left(\dfrac{z}{\beta}\right)}=1-\frac{\eta}{\ln\left(\dfrac{z}{\bar{z}}\right)}>1,\ \ \text{for } z<\bar{z}\equiv\beta e^{\frac{\eta}{\sigma-1}}$$

which is strictly increasing in $z\in(0,\bar{z})$ with the range $(1,\infty)$, and hence satisfying the strong 2nd Law. Homothetic symmetric translog is a special case of this family, where $\eta=1$.[30] CES is the limit case of this family, as $\eta\to\infty$, while holding $\beta>0$ and $\sigma>1$ fixed, with

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[29]This is so even though CES has no choke price, because the choke price goes to infinity as one takes the limit in these families.

[30]To see this, eq. (19') of Feenstra (2003) gives the expression for the market share for each product under translog as $\frac{p(\omega)}{P(\mathbf{p})}\frac{\partial P(\mathbf{p})}{\partial p(\omega)}=\frac{1}{N}-\gamma'\left[\ln p(\omega)-\frac{1}{N}\int_{\Omega}\ln p(\omega')\,d\omega'\right], (\gamma'>0)$, where N is the measure of Ω. This can be rewritten

$$z < \bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}} \to \infty;$$

$$s(z) = \gamma\left(1 - \frac{\sigma-1}{\eta}\ln\left(\frac{z}{\beta}\right)\right)^{\eta} \to \gamma\left(\frac{z}{\beta}\right)^{1-\sigma};$$

$$\zeta(z) = 1 + \frac{\sigma-1}{1 - \frac{\sigma-1}{\eta}\ln\left(\frac{z}{\beta}\right)} \to \sigma.$$

Because $\ln(\zeta(z)-1) = \ln\eta - \ln(\ln(\bar{z}/z))$ is convex for $z \in (\bar{z}/e\,, \bar{z})$, the log-concavity condition in Proposition 2 fails. It is thus necessary to go through a calculation explicitly. From

$$\frac{L}{F} = \frac{\zeta(\underline{z}^m)}{s(\underline{z}^m)} = \frac{\left(\ln(\bar{z}/\underline{z}^m)\right)^{-\eta} + \eta\left(\ln(\bar{z}/\underline{z}^m)\right)^{-\eta-1}}{\gamma\left(\frac{\sigma-1}{\eta}\right)^{\eta}},$$

where $\underline{z}^m$ is strictly increasing in $L/F \in (0,\infty)$ with the range $(0,\bar{z})$. Using

$$\underline{z}^c = \underline{z}^m\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right] = \frac{\eta\underline{z}^m}{\eta + \ln(\bar{z}/\underline{z}^m)},$$

we obtain

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \left[\frac{\ln(\bar{z}/\underline{z}^c)}{\ln(\bar{z}/\underline{z}^m)}\right]^{\eta} = \left[\frac{\ln(\underline{z}^m/\bar{z}) + \ln\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]}{\ln(\underline{z}^m/\bar{z})}\right]^{\eta}$$

$$= \left[1 + \frac{1}{\eta}\ln\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]^{1-\zeta(\underline{z}^m)}\right]^{\eta} < \left(1 + \frac{1}{\eta}\right)^{\eta} < e.$$

Because θ is increasing in $\zeta(\underline{z}^m)$, which is increasing in $\underline{z}^m$, θ is strictly increasing in $\underline{z}^m$ and hence strictly increasing in L/F with the range, $1 < \theta < (1+1/\eta)^{\eta}$. From this, we can conclude that the condition for the stable steady state always holds if $\delta(\theta-1) < \delta[(1+1/\eta)^{\eta}-1] < 1$. If $\delta[(1+1/\eta)^{\eta}-1] > 1 > \delta^2[(1+1/\eta)^{\eta}-1]$, a stable period-2 cycle merges for a sufficiently high L/F. If $\delta^2[(1+1/\eta)^{\eta}-1] > 1$, an increase in L/F first leads to the emergence of a stable period-2 cycle, which then becomes unstable, and leads to the emergence of a chaotic attractor. Note that the existence of endogenous fluctuations requires the

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as $\frac{p(\omega)}{P(\mathbf{p})}\frac{\partial P(\mathbf{p})}{\partial p(\omega)} = -\gamma'\ln\left(\frac{p(\omega)}{A(\mathbf{p})}\right)$, with $\ln A(\mathbf{p}) \equiv \frac{1}{\gamma N} + \frac{1}{N}\int_{\Omega}\ln p(\omega')\,d\omega'$ and $s(z) \equiv \gamma'\max\{\ln(1/z)\,,0\}$, which can be obtained by setting $\eta = 1$ and normalizing $\gamma' = \gamma(\sigma-1)$ and $\beta^{\sigma-1} = e$.

$\eta > 1$. Furthermore, for $\delta > (e-1)^{-1} \approx 0.582$, it is more likely for a large L/F to generate endogenous fluctuations (even chaotic fluctuations for $\delta > (e-1)^{-1/2} \approx 0.763$), as η becomes larger, i.e., when it is closer to the limit case of CES within this family. This occurs because the upper bound of $\theta, (1+1/\eta)^{\eta}$, is independent of $\sigma > 1$ and $(1+1/\eta)^{\eta} \to e$, as we approach to the CES limit, $\eta \to \infty$, and yet, in the CES limit, $\theta = (1-1/\sigma)^{1-\sigma} < e$. In other words, the qualitative properties of the dynamical system change *discontinuously* with an arbitrarily small departure from CES, even though the underlying demand system converges to CES; the destabilizing effects of the market size/innovation cost ratio through the procompetitive effect become *amplified* as the demand system approaches to the CES limit.

Example 2: Constant Pass-Through[31]

$$s(z) = \begin{cases} \gamma\left[\sigma - (\sigma-1)\left(\dfrac{z}{\beta}\right)^{\Delta}\right]^{1/\Delta} = \gamma\sigma^{\frac{1}{\Delta}}\left[1 - \left(\dfrac{z}{\bar{z}}\right)^{\Delta}\right]^{1/\Delta} & \text{for } z < \bar{z} \equiv \beta\left(\dfrac{\sigma}{\sigma-1}\right)^{1/\Delta} \\ 0 & \text{for } z \geq \bar{z} \equiv \beta\left(\dfrac{\sigma}{\sigma-1}\right)^{1/\Delta} \end{cases}$$

with constant parameters, $\gamma, \beta, \Delta > 0$ and $\sigma > 1$. Then,

$$\zeta(z) = \frac{1}{1 - \left(1 - \frac{1}{\sigma}\right)\left(\frac{z}{\beta}\right)^{\Delta}} = \frac{1}{1 - \left(\frac{z}{\bar{z}}\right)^{\Delta}} > 1, \text{for } z < \bar{z} \equiv \beta\left(\frac{\sigma}{\sigma-1}\right)^{1/\Delta}$$

which is strictly increasing in $z \in (0, \bar{z})$ with the range $(1, \infty)$, and hence satisfying the strong 2nd Law.

From the pricing rule, $z_t^c, z_t^m \in (0, \bar{z})$ satisfy:

$$z_t^c = z_t^m\left[1 - \frac{1}{\zeta(z_t^m)}\right] = z_t^m\left(\frac{z_t^m}{\bar{z}}\right)^{\Delta} \Rightarrow z_t^m = (\bar{z})^{\frac{\Delta}{1+\Delta}}(z_t^c)^{\frac{1}{1+\Delta}}$$

$$\Rightarrow \ln p_t^m = \frac{\Delta}{1+\Delta}\ln(A(\mathbf{p}_t)\bar{z}) + \frac{1}{1+\Delta}\ln(\psi)$$

so that this family is characterized by the constant pass-through rate, $0 < 1/(1+\Delta) < 1$. CES is the limit case of this family as $\Delta \to 0$, while holding $\beta > 0$ and $\sigma > 1$ fixed, with

$$\bar{z} \equiv \beta\left(\frac{\sigma}{\sigma-1}\right)^{1/\Delta} \to \infty;$$

[31]In Matsuyama and Ushchev (2020b), we proposed three parametric families of demand systems, all featuring a constant pass-through rate. One of them is within the class of H.S.A. This is its symmetric version.

$$\zeta(z) = \frac{1}{1 - \left(1 - \frac{1}{\sigma}\right)\left(\frac{z}{\beta}\right)^{\Delta}} \to \sigma$$

and, using l'Hôpital's rule:

$$\lim_{\Delta \to 0} \ln \frac{s(z)}{\gamma} = \lim_{\Delta \to 0} \frac{\ln\left[\sigma - (\sigma - 1)\left(\frac{z}{\beta}\right)^{\Delta}\right]}{\Delta} = \lim_{\Delta \to 0} \frac{(1-\sigma)\left(\frac{z}{\beta}\right)^{\Delta} \ln z}{\sigma - (\sigma - 1)\left(\frac{z}{\beta}\right)^{\Delta}} = (1 - \sigma)\ln z,$$

and hence

$$s(z) = \gamma\left[\sigma - (\sigma - 1)\left(\frac{z}{\beta}\right)^{\Delta}\right]^{1/\Delta} \to \gamma z^{1-\sigma}.$$

This family satisfies (A2), but $\zeta(z) - 1$ is not log-concave, so we cannot use Proposition 2 to show that θ is increasing in L/F. To calculate θ explicitly,

$$\underline{z}^c = \underline{z}^m\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right] = \underline{z}^m\left(\frac{\underline{z}^m}{\bar{z}}\right)^{\Delta} \Rightarrow \frac{\underline{z}^c}{\bar{z}} = \left(\frac{\underline{z}^m}{\bar{z}}\right)^{1+\Delta};$$

$$\frac{s(\underline{z}^m)}{\zeta(\underline{z}^m)}\frac{L}{F} \equiv 1 \Rightarrow \gamma\sigma^{\frac{1}{\Delta}}\frac{L}{F}\left[1 - \left(\frac{\underline{z}^m}{\bar{z}}\right)^{\Delta}\right]^{1+1/\Delta} = 1 \Rightarrow \left(\frac{\underline{z}^c}{\bar{z}}\right)^{\frac{\Delta}{1+\Delta}} = \left(\frac{\underline{z}^m}{\bar{z}}\right)^{\Delta} = 1 - \left(\frac{F}{\gamma\sigma^{\frac{1}{\Delta}}L}\right)^{\frac{\Delta}{1+\Delta}}$$

for $F/L < \gamma\sigma^{\frac{1}{\Delta}}$. In this range, $\underline{z}^c$ and $\underline{z}^m$ are both strictly increasing in L/F. Since

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \left[\frac{1 - \left(\frac{\underline{z}^c}{\bar{z}}\right)^{\Delta}}{1 - \left(\frac{\underline{z}^m}{\bar{z}}\right)^{\Delta}}\right]^{1/\Delta} = \left[\frac{1 - \left(\frac{\underline{z}^m}{\bar{z}}\right)^{\Delta(1+\Delta)}}{1 - \left(\frac{\underline{z}^m}{\bar{z}}\right)^{\Delta}}\right]^{1/\Delta}$$

is also strictly increasing in $\underline{z}^m$,[32] θ is strictly increasing in L/F with the range, $1 < \theta < (1 + \Delta)^{1/\Delta}$. The upper bound of θ, $(1 + \Delta)^{1/\Delta}$, is decreasing in Δ and $(1 + \Delta)^{1/\Delta} \lesseqgtr 2$ for $\Delta \gtreqless 1$ and $(1 + \Delta)^{1/\Delta} \to e$, as $\Delta \to 0$. From this, we can conclude that the steady state is always stable if $\delta(\theta - 1) < \delta\left[(1 + \Delta)^{1/\Delta} - 1\right] < 1$. If $\delta\left[(1 + \Delta)^{1/\Delta} - 1\right] > 1 > \delta^2\left[(1 + \Delta)^{1/\Delta} - 1\right]$, which requires $\Delta < 1$, a stable period-2 cycle emerges for a sufficiently high L/F. If $\delta^2\left[(1 + \Delta)^{1/\Delta} - 1\right] > 1$, an increase in L/F first causes the loss of the stability of the

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[32] To see this, let $\xi \equiv \left(\underline{z}^m/\bar{z}\right)^{\Delta}$ so that $\theta \equiv \theta(\xi) = \left(\frac{1-\xi^{(1+\Delta)}}{1-\xi}\right)^{1/\Delta}$. Then, $\frac{d}{d\xi}\left(\frac{1-\xi^{(1+\Delta)}}{1-\xi}\right) = \frac{\Delta\xi^{(1+\Delta)} - (1+\Delta)\xi^{\Delta} + 1}{(1-\xi)^2} \equiv \frac{N(\xi)}{(1-\xi)^2} > 0$ for $0 < \xi < 1$, because $N(1) = 0$ and $N'(\xi) = (1 + \Delta)\Delta(\xi - 1)\xi^{\Delta - 1} < 0$ for $0 < \xi < 1$.

steady state, which leads to the emergence of a stable period-2 cycle. A further increase in L/F then causes the loss of the stability of the period-2 cycle, which leads to the emergence of a chaotic attractor. Note that the existence of endogenous fluctuations requires the (constant) pass-through rate, $1/(1+\Delta)$, to be greater than one half. Furthermore, for $\delta > (e-1)^{-1} \approx 0.582$, it is more likely for a large L/F to generate endogenous fluctuations (even chaotic fluctuations for $\delta > (e-1)^{-1/2} \approx 0.763$), as Δ becomes smaller and hence the pass-through rate, $1/(1+\Delta)$, becomes closer to one, i.e., when it is closer to the limit case of CES within this family. This occurs because the upper bound of θ, $(1+\Delta)^{1/\Delta}$, is independent of $\sigma > 1$ and $(1+\Delta)^{1/\Delta} \to e$, as we approach to the CES limit, $\Delta \to 0$, and yet, in the CES limit, $\theta = (1-1/\sigma)^{1-\sigma} < e$. In other words, similar to the generalized translog family, the qualitative properties of the dynamical system under the constant pass-through family change *discontinuously* with an arbitrarily small departure from CES, even though the underlying demand system converges to CES; the destabilizing effects of the market size/innovation cost ratio through the procompetitive effect become *amplified* as the demand system approaches to the CES limit.

7. A Multi-Market Extension

Let us now consider some cross-sectional implications in a multi-market/sector extension. Thanks to the homotheticity of the H.S.A. demand systems, such an extension is straightforward. Imagine that there are J markets or sectors, indexed by $j = 1,2,\dots,J$. Each produces the single consumption good, j, whose market size is L_j, by assembling j-specific intermediate inputs in Ω_j, with CRS technology that belongs to the H.S.A. class, characterized by $s_j(\cdot)$; with the innovation cost, F_j, and the survival rate, δ_j. Market size, L_j, is exogenously fixed. This could be justified, for example, by assuming that there are identical households, which collectively supply L units of labor, and their preferences are given by Cobb-Douglas of the following form, $\sum_{j=1}^{J} \beta_j \ln X_j$ with $\sum_{j=1}^{J} \beta_j = 1$. Then, $L_j = \beta_j L$. Alternatively, there may be J different types of consumers, with L_j being the total income of type-j consumers, who consume only type-j consumption good. Here "types" may be their "tastes" or "cities they live." Then, the dynamics of innovations in different markets/sectors are decoupled, with each following

$$n_{j,t+1} = f_j(n_{j,t}) \equiv \begin{cases} f_{jL}(n_{j,t}) \equiv & \delta_j\left(\theta_j + (1-\theta_j)n_{j,t}\right) \quad for\ n_{j,t} < 1 \\ f_{jH}(n_{j,t}) \equiv & \delta_j n_{j,t} \quad for\ n_{j,t} > 1 \end{cases}$$

where θ_j is given by

$$\theta_j \equiv \frac{s_j\left(\underline{z}_j^m\left[1-\frac{1}{\zeta_j\left(\underline{z}_j^m\right)}\right]\right)}{s_j\left(\underline{z}_j^m\right)} > 1; \ \frac{\zeta_j\left(\underline{z}_j^m\right)}{s_j\left(\underline{z}_j^m\right)} = \frac{1}{s_j\left(\underline{z}_j^m\right)}\left[1-\frac{\underline{z}_j^m s_j'\left(\underline{z}_j^m\right)}{s_j\left(\underline{z}_j^m\right)}\right] \equiv \frac{L_j}{F_j}.$$

This allows us to do market-by-market analysis.

To consider cross-sectional implications, let us think of J metropolitan areas, which differ only in market size, L_j, so that we can index them in such a way that $L_1 < L_2 < \cdots < L_J$. Then, under either of the two complementary sets of sufficient conditions discussed above, $\theta_1 < \theta_2 < \cdots \theta_J$. Furthermore, since the steady state n^* is increasing in θ, $n_1^* < n_2^* < \cdots < n_J^*$. Thus, over the long run, metropolitan areas with larger markets have more innovation/entry activities (even in per capita term, if market size is due to the population size). However, the dynamics are more volatile in larger markets. This implies, among others, that the sales of each product, conditional on surviving idiosyncratic obsolescence shocks, fluctuates more in larger markets.[33] Furthermore, the dynamics may converge to the steady state and innovation occurs steadily in smaller markets, while they may converge to periodic cycles or even chaotic attractors so that innovations occur only intermittently in larger markets. Hence, innovation/entry activities are not always higher in larger markets than in smaller markets.[34]

8. Concluding Remarks

In this paper, we have studied how market size affects the patterns of fluctuations in the dynamics of innovation. Previous models of endogenous innovation cycles were silent on this issue, because a change in market size alters the amplitude of fluctuations but not the nature of fluctuations. This is due to the ubiquitous assumption of CES homothetic demand system, under which monopolistically competitive firms sell their products at an exogenous markup rate. This feature stands at odds with ample empirical evidence for the procompetitive effect of entry and market size. In the presence of such procompetitive effect, a larger market invites more firms to

[33]If the innovating firms remain the sole producers of their products even after they have lost the monopoly power, which may be justified if the competitive fringes have to pay an arbitrarily small fixed cost to produce, this also means that the revenue of firms fluctuate more in larger markets. This is in line with the empirical evidence by Gaubert (2014, Ch.1) in cross-sections of the French metropolitan areas.

[34]In a different context, Vives (2008) shows that, if firms have access to process innovations, which lower their marginal costs, larger market size could lead to more process innovations, not necessarily more entry of new firms.

enter. As a result, firms face more elastic demand, which forces them to set their prices at lower markup rates. Such competitive pressures make innovations more sensitive to changing market environments, thereby causing instability in the dynamics of innovation. To capture this intuition, we extended the Judd model of endogenous innovation cycles; we allowed for the procompetitive effect by replacing the assumption of CES demand system with a more general homothetic demand system, H.S.A., which contains both CES and homothetic translog as special cases. We show that a larger market size/innovation cost ratio has destabilizing effects in the dynamics of innovation through the procompetitive effect under two complementary sets of sufficient conditions; i) when the price elasticity is "not too convex" in price; and ii) when the demand system belongs to two parametric families of "generalized translog" or "constant pass-through," each of which features the choke price and yet contains CES as a limit case. This allows us to use them to check the robustness of the results under CES. One might think *a priori* that, as the demand system approaches to the CES limit within each family, the destabilizing effects would become smaller and converge to zero. It turns out, however, that they become *amplified* as the demand system approaches to the CES limit. In other words, the qualitative properties of the dynamics change *discontinuously* with an arbitrarily small departure from CES within each family, even though the underlying demand system converges to CES. This provides a caution for using the CES demand system as an approximation even if the estimated pass-through rate is arbitrarily close to one. We also discussed cross-sectional implications in a multi-market extension. Because innovation/entry activities fluctuate more in larger markets, they are not always higher in larger markets than in smaller markets. Furthermore, the sale of each product, conditional on surviving idiosyncratic obsolescence shocks, is more volatile in larger markets.

One might wonder how the analysis needs to be modified if we use, instead of H.S.A., HDIA and HIIA, the two alternative classes of homothetic demand systems, studied in Matsuyama and Ushchev (2020a). A preliminary investigation reveals that the equilibrium trajectory is obtained by iterating a one-dimensional piecewise smooth map with two branches, nonlinear and decreasing in the lower branch and linear and increasing in the upper branch. In other words, the slope of the lower branch, determined by the delayed impact of innovation, is no longer a constant. Thus, the map is no longer analytically solvable and needs to be solved numerically. Nevertheless, we conjecture that many features of a skewed-V map are preserved, and qualitative insights carry over. This is because a skewed-V map can be used as a linear approximation (*a normal form*, in the language of the dynamical system theory) for such piecewise smooth maps; see Avrutin et.al. (2019, chapter 5.5); for an application of this technique in economics, see Matsuyama, Sushko, and Gardini (2016; Fig.8).[35]

Beyond this application, H.S.A. demand systems should provide a useful alternative for the ubiquitous CES demand system in monopolistic competition models. Being tractable and yet capable of accommodating the procompetitive effect, they can capture mechanisms that cannot be captured by the CES assumption.[36] In particular, the two parametric families in Section 6, "generalized translog" and "constant pass-through," should find many applications.

[35]One might also wonder how market size could affect the nature of fluctuations through the procompetitive effect in other models of endogenous innovation cycles, particularly in the implementation cycle model of Shleifer (1986). First, unlike the Judd model, which generates endogenous fluctuation along the unique equilibrium path, the Shleifer model generates endogenous fluctuations through expectational indeterminacy and multiple equilibria. There always exists an equilibrium path that does not fluctuate in the absence of exogenous shocks. We conjecture that introducing the procompetitive effect would not affect such a non-fluctuating equilibrium path. Second, as already explained in footnote 9, Shleifer's mechanism, in contrast to Judd's, hinges on the strategic complementarity in innovation, that is, the monopoly profit earned by an innovator, goes up if more innovations occur at the same time. The procompetitive effect, which causes the markup rate to go down, would weaken the strategic complementarity, thereby reducing the likelihood of multiple equilibria, and possibly eliminating fluctuating equilibrium paths. For this reason, we conjecture that larger market size, and resulting competitive pressures have stabilizing effects in Shleifer's model. If this conjecture is correct, it suggests that studying market size and volatility of innovation/entry activities provides a way to test these two alternative mechanisms of generating endogenous innovation cycles.
[36]For recent applications of H.S.A., see Grossman, Helpman, and Lhuillier (2021) and Matsuyama and Ushchev (working in progress).

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Appendix A: Kinked Demand System: An Example

Consider the case where $s(\cdot)$ has a kink, so that it is only piecewise continuously differentiable. In this case, $\zeta(z)$ is discontinuous at the kink. Furthermore, the profit maximizing value of z_t^m may occur at a discontinuity point of $\zeta(z)$, where $\zeta(\cdot)$ jumps up, i.e., $s'(\cdot)$ jumps down or equivalently, $|s'(\cdot)|$ jumps up. If so, it is characterized by:

$$\lim_{z\uparrow z^d} z\left[1-\frac{1}{\zeta(z)}\right] < z_t^c < \lim_{z\downarrow z^d} z\left[1-\frac{1}{\zeta(z)}\right] < z_t^m = z^d; \lim_{z\uparrow z^d}\frac{\zeta(z)}{s(z)} < \frac{L}{F} < \lim_{z\downarrow z^d}\frac{\zeta(z)}{s(z)}$$

by denoting the discontinuity point by z^d. The following example illustrates such a case.

Example A: kinked CES: For $0 < \varepsilon < \sigma - 1$,

$$s(z) = \begin{cases} z^{1-(\sigma-\varepsilon)} & \text{if} \quad z \le 1 \\ z^{1-\sigma} & \text{if} \quad z \ge 1 \end{cases} \Rightarrow \zeta(z) = \begin{cases} \sigma-\varepsilon & \text{if} \quad z<1 \\ [\sigma-\varepsilon,\sigma] & \text{if} \quad z=1 \\ \sigma & \text{if} \quad z>1 \end{cases}$$

$$\Rightarrow \frac{\zeta(z)}{s(z)} = \begin{cases} (\sigma-\varepsilon)z^{\sigma-\varepsilon-1} & \text{if} \quad z<1 \\ [\sigma-\varepsilon,\sigma] & \text{if} \quad z=1. \\ \sigma z^{\sigma-1} & \text{if} \quad z>1 \end{cases}$$

For $L/F < \sigma-\varepsilon$, we have $\underline{z}^c < \underline{z}^m < z^d = 1$ with

$$\underline{z}^c = \left(\frac{L/F}{\sigma-\varepsilon}\right)^{\frac{1}{(\sigma-\varepsilon)-1}}\left[1-\frac{1}{\sigma-\varepsilon}\right] = \underline{z}^m\left[1-\frac{1}{\sigma-\varepsilon}\right] < \underline{z}^m = \left(\frac{L/F}{\sigma-\varepsilon}\right)^{\frac{1}{(\sigma-\varepsilon)-1}} < 1$$

$$\Longrightarrow \theta = \left(1-\frac{1}{\sigma-\varepsilon}\right)^{1-(\sigma-\varepsilon)}$$

For $\sigma-\varepsilon < L/F < \sigma$, we have $\underline{z}^c < \underline{z}^m = z^d = 1$ with,

$$\underline{z}^c = 1-\frac{F}{L} < \underline{z}^m = 1 \Longrightarrow \theta = \left(1-\frac{F}{L}\right)^{1-(\sigma-\varepsilon)}.$$

For $\sigma < L/F < \sigma\left(1-\frac{1}{\sigma}\right)^{1-\sigma}$, we have $\underline{z}^c < z^d = 1 < \underline{z}^m$ with,

$$\underline{z}^c = \left(\frac{L}{\sigma F}\right)^{\frac{1}{\sigma-1}}\left[1-\frac{1}{\sigma}\right] = \underline{z}^m\left[1-\frac{1}{\sigma}\right] < 1 < \underline{z}^m = \left(\frac{L}{\sigma F}\right)^{\frac{1}{\sigma-1}} \Longrightarrow \theta = \left(\frac{L}{\sigma F}\right)^{\frac{\varepsilon}{\sigma-1}}\left[1-\frac{1}{\sigma}\right]^{1-\sigma+\varepsilon}.$$

For $L/F > \sigma\left(1-\frac{1}{\sigma}\right)^{1-\sigma}$ we have $z^d = 1 < \underline{z}^c < \underline{z}^m$ with,

$$1 < \underline{z}^c = \left(\frac{L}{\sigma F}\right)^{\frac{1}{\sigma-1}}\left[1-\frac{1}{\sigma}\right] = \underline{z}^m\left[1-\frac{1}{\sigma}\right] < \underline{z}^m = \left(\frac{L}{\sigma F}\right)^{\frac{1}{\sigma-1}} \Longrightarrow \theta = \left(1-\frac{1}{\sigma}\right)^{1-\sigma}.$$

Hence, θ is strictly increasing in L/F for $\sigma - \varepsilon < L/F < \sigma\left(1-\frac{1}{\sigma}\right)^{1-\sigma}$ and constant otherwise.

Appendix B: An example, showing that $\boldsymbol{\theta}$ can be arbitrarily large without (A2).

Example B:

$$s(z) = \exp\left[\int_1^z \frac{1-\zeta(\tau)}{\tau}d\tau\right],$$

where $\zeta(z)$ is given by

$$1-\frac{1}{\zeta(z)} = \begin{cases} 1 - A^{-\beta}(1-A^{1-\alpha})z^{\beta} & if \quad z \le A \\ Az^{-\alpha} & if \quad z > A \end{cases}$$

where $\alpha \in (0,1)$ and $A \in (0,1)$ and

$$\beta \equiv \frac{\alpha A^{1-\alpha}}{1-A^{1-\alpha}} > 0$$

By construction, $\zeta(z) > 1$, and continuously differentiable. Hence $s(z)$ is twice-continuously differentiable and strictly decreasing with $\lim_{z\to 0}s(z) = \infty$, $\lim_{z\to\infty}s(z) = 0$, and $s(1) = 1$. Furthermore,

$$z\left[1-\frac{1}{\zeta(z)}\right] = \begin{cases} z - A^{-\beta}(1-A^{1-\alpha})z^{1+\beta} & if \quad z \le A \\ Az^{1-\alpha} & if \quad z > A \end{cases}$$

is strictly increasing. Hence, (A1) holds, even though $\zeta(z)$ is strictly decreasing, and hence (A2) is violated. From iii) in Lemma, $s(z)/\zeta(z)$ is strictly decreasing. For $F/L = 1 - A$, the unique solution of

$$\frac{s(\underline{z}^m)}{\zeta(\underline{z}^m)}\frac{L}{F} = 1$$

is given by $\underline{z}^m = 1$, from which

$$\underline{z}^m = 1 > \underline{z}^c = \underline{z}^m\left[1-\frac{1}{\zeta(\underline{z}^m)}\right] = A,$$

from which

$$1 - \frac{1}{\zeta(\underline{z}^m)} = A; 1 - \frac{1}{\zeta(\underline{z}^c)} = A^{1-\alpha}$$

Because $s(z)/\zeta(z)$ is strictly decreasing, this implies

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} > \frac{\zeta(\underline{z}^c)}{\zeta(\underline{z}^m)} = \frac{1-A}{1-A^{1-\alpha}},$$

where the RHS becomes arbitrarily large as $\alpha \to 1$.

Thus, without (A2), the unique attractor of dynamical system eq.(24) could be a stable cycle of any positive number of periods or a chaotic attractor with any positive number of cyclic intervals, depending on θ and δ.

Appendix C: What might happen when (A1) is violated

Now, let us consider what might happen when (A1) is violated so that

$$1 - \zeta(z) > \frac{z\zeta'(z)}{\zeta(z)}$$

for some $z \in (0, \bar{z})$. Then, from Lemma,

i) $z(1 - 1/\zeta(z))$ is strictly decreasing at such $z \in (0, \bar{z})$.

ii) For some $z^c \in (0, \bar{z})$, $\pi(z) \equiv (1 - z^c/z)s(z)$ has multiple peaks in $z \in (z^c, \bar{z})$.

iii) $s(z)/\zeta(z)$ is strictly increasing at such $z \in (0, \bar{z})$.

In this case, we need to worry about the possibility that there may be more than one relative price, z_t^m, that maximizes the profit of monopolists in equilibrium. If two such relative prices, z_{1t}^m and $z_{2t}^m > z_{1t}^m$, exist, they must satisfy the following conditions:

$$z_{1t}^m \left[1 - \frac{1}{\zeta(z_{1t}^m)}\right] = z_{2t}^m \left[1 - \frac{1}{\zeta(z_{2t}^m)}\right] = z_t^c,$$

$$\frac{s(z_{1t}^m)}{\zeta(z_{1t}^m)} = \frac{s(z_{2t}^m)}{\zeta(z_{2t}^m)} = \frac{F}{L}$$

where both z_{1t}^m and $z_{2t}^m > z_{1t}^m$ must satisfy the SOC, which means that they are at an increasing segment of $z(1 - 1/\zeta(z))$ and at a decreasing segment of $s(z)/\zeta(z)$. Furthermore, the budget constraint implies

$$V_{1t}^m s(z_{1t}^m) + V_{2t}^m s(z_{2t}^m) + V_t^c s(z_t^c) = 1,$$

where $V_{1t}^m > 0$ innovators/monopolists select z_{1t}^m and $V_{2t}^m > 0$ innovators/monopolists select z_{2t}^m.

For this to happen,

First, there must exist z_1^m and $z_2^m > z_1^m$ that solve the following two equations:

$$z_1^m \left[1 - \frac{1}{\zeta(z_1^m)}\right] = z_2^m \left[1 - \frac{1}{\zeta(z_2^m)}\right]$$

$$\frac{s(z_1^m)}{\zeta(z_1^m)} = \frac{s(z_2^m)}{\zeta(z_2^m)}$$

Second, the value of F/L must coincide with the common value of $s(z)/\zeta(z)$ at z_1^m and z_2^m:

$$\frac{s(z_1^m)}{\zeta(z_1^m)} = \frac{s(z_2^m)}{\zeta(z_2^m)} = \frac{F}{L}$$

Third, the value of V_t^c must satisfy

$$V_t^c s(z^c) < 1,$$

where z^c is given by the common value of $z(1 - 1/\zeta(z))$ at z_1^m and z_2^m:

$$z^c = z_1^m \left[1 - \frac{1}{\zeta(z_1^m)}\right] = z_2^m \left[1 - \frac{1}{\zeta(z_2^m)}\right]$$

Then, any combination of $V_{1t}^m > 0$ and $V_{2t}^m > 0$ satisfying

$$V_{1t}^m s(z_1^m) + V_{2t}^m s(z_2^m) = 1 - V_t^c s(z^c) > 0.$$

can be an equilibrium. This means that the total innovation, $V_t^m = V_{1t}^m + V_{2t}^m$ can be any value in

$$\frac{1}{s(z_1^m)} - \frac{s(z^c)}{s(z_1^m)} V_t^c \leq V_t^m \leq \frac{1}{s(z_2^m)} - \frac{s(z^c)}{s(z_2^m)} V_t^c$$

Or

$$\frac{1}{s(z_1^m)} + (1 - \theta_1) V_t^c \leq V_t^c + V_t^m \leq \frac{1}{s(z_1^m)} + (1 - \theta_2) V_t^c$$

where

$$\theta_1 \equiv \frac{s(z^c)}{s(z_1^m)} < \theta_2 \equiv \frac{s(z^c)}{s(z_2^m)}$$

From this, the dynamical system of $n_t = s(z^c) V_t^c$ becomes:

$$\delta \max\{\theta_1 + (1 - \theta_1) n_t, n_t\} \leq n_{t+1} \leq \delta \max\{\theta_2 + (1 - \theta_2) n_t, n_t\}$$

Hence, the dynamical system becomes ill-defined (or allow for a continuum of paths).

However, this can occur only if the value of F/L happens to be equal to:

$$\frac{s(z_1^m)}{\zeta(z_1^m)} = \frac{s(z_2^m)}{\zeta(z_2^m)} = \frac{F}{L}$$

where z_1^m and $z_2^m > z_1^m$ solve the following two equations:

$$z_1^m \left[1 - \frac{1}{\zeta(z_1^m)}\right] = z_2^m \left[1 - \frac{1}{\zeta(z_2^m)}\right]$$
$$\frac{s(z_1^m)}{\zeta(z_1^m)} = \frac{s(z_2^m)}{\zeta(z_2^m)}$$

Hence, generically, this can occur only for a finite number of particular values of F/L. And when a change in F/L crosses such a particular value from below, z^m jumps from z_1^m to z_2^m and θ jumps from $\theta_1 \equiv s(z^c)/s(z_1^m)$ to $\theta_2 \equiv s(z^c)/s(z_2^m)$.

To illustrate this, consider the following example (although this example implies that $\zeta(z)$ is discontinuous)

Example C: kinked CES: For $0 < \varepsilon < \sigma - 1$, define the market share function as follows:

$$s(z) = \max\{z^{1-\sigma}, z^{1-(\sigma-\varepsilon)}\}.$$

The corresponding zeta-function is well defined for all $z \neq 1$, and is given by:

$$\zeta(z) = \begin{cases} \sigma, & z < 1, \\ \sigma - \varepsilon, & z > 1. \end{cases}$$

In what follows, $\pi(z)$ will denote the profit function, while $\pi^*(z^c)$ will denote the maximum value of $\pi(z)$ as a function of z^c.

Lemma A1.

i) If $z^c \leq 1 - 1/(\sigma - \varepsilon)$, then $\pi(z)$ is single peaked, and its maximizer is smaller than 1.

ii) If $1 - 1/(\sigma - \varepsilon) < z^c < 1 - 1/\sigma$, then $\pi(z)$ has two local maximizers, one smaller than 1 and the other greater than 1;

iii) If $z^c \geq 1 - 1/\sigma$, then $\pi(z)$ is single peaked, and its maximizer is greater than 1.

Proof: It is readily verified that $\pi(z)$ can be represented as follows;

$$\pi(z) = \max\{\pi_1(z), \pi_2(z)\},$$

where

$$\pi_1(z) \equiv \left(1 - \frac{z^c}{z}\right) z^{1-\sigma}, \qquad \pi_2(z) \equiv \left(1 - \frac{z^c}{z}\right) z^{1-(\sigma-\varepsilon)}.$$

Both $\pi_1(z)$ and $\pi_2(z)$ are single-peaked. Furthermore, evaluating the derivatives of $\pi_1(z)$ and $\pi_2(z)$ at $z = 1$ yields:

$$\pi_1'(1) = z^c + (1 - z^c)(1 - \sigma), \qquad \pi_2'(1) = z^c + (1 - z^c)[1 - (\sigma - \varepsilon)],$$

which yields:

$$z^c \le 1-\frac{1}{\sigma-\varepsilon} \Leftrightarrow \pi_1'(1) < 0 \text{ and } \pi_2'(1) \le 0 \Longrightarrow \text{i)}$$

$$1-\frac{1}{\sigma-\varepsilon} < z^c < 1-\frac{1}{\sigma} \Leftrightarrow \pi_1'(1) < 0 < \pi_2'(1) \Longrightarrow \text{ii)}$$

$$z^c \ge 1-\frac{1}{\sigma} \Leftrightarrow \pi_1'(1) \ge 0 \text{ and } \pi_2'(1) > 0 \Longrightarrow \text{iii)}$$

This completes the proof. ■

Lemma A2. There exists a unique value $\tilde{z} \in \left(1-\frac{1}{\sigma-\varepsilon}, 1-\frac{1}{\sigma}\right)$, such that:

i) If $z^c < \tilde{z}$, then $z_1^m \equiv \frac{z^c}{1-1/\sigma} < 1$ is the unique global profit maximizer;

ii) If $z^c > \tilde{z}$, then $z_2^m \equiv \frac{z^c}{1-1/(\sigma-\varepsilon)} > 1$ is the unique global profit maximizer;

iii) If $z^c = \tilde{z}$, then both $z_1^m < 1$ and $z_2^m > 1$ are global profit maximizers.

Proof. We start with proving part iii). It is readily verified that

$$\pi^*(z^c) = \max\{\pi_1^*(z^c), \pi_2^*(z^c)\},$$

where

$$\pi_1^*(z^c) \equiv \max_{z\ge 0} \pi_1(z) = \frac{1}{\sigma}\left(1-\frac{1}{\sigma}\right)^{\sigma-1} (z^c)^{1-\sigma},$$

$$\pi_2^*(z^c) \equiv \max_{z\ge 0} \pi_2(z) = \frac{1}{\sigma-\varepsilon}\left(1-\frac{1}{\sigma-\varepsilon}\right)^{\sigma-\varepsilon-1} (z^c)^{1-(\sigma-\varepsilon)}.$$

Thus, $z^c = \tilde{z}$ must be a solution to the following equation:

$$\pi_1^*(z^c) = \pi_2^*(z^c).$$

Because $\pi_1^*(z^c)$ and $\pi_2^*(z^c)$ are power functions with different exponents, this equation has a unique positive solution $z^c = \tilde{z}$, where $\tilde{z}$ is given by:

$$\tilde{z} \equiv \left[\frac{(\sigma-\varepsilon)\left(1-\frac{1}{\sigma-\varepsilon}\right)^{1-(\sigma-\varepsilon)}}{\sigma\left(1-\frac{1}{\sigma}\right)^{1-\sigma}}\right]^{\frac{1}{\varepsilon}}.$$

In this case, $\pi(z)$ has two global maximizers given by:

$$z_1^m \equiv \frac{\tilde{z}}{1-1/\sigma}, \qquad z_2^m \equiv \frac{\tilde{z}}{1-1/(\sigma-\varepsilon)}.$$

This proves part iii).

When $z^c < \tilde{z}$, we have $\pi_1^*(z^c) > \pi_2^*(z^c) \Longrightarrow \pi_1^*(z^c) = \pi^*(z^c)$, hence $z_1^m < 1$ is a unique global maximizer of $\pi(z)$, which proves part i). Likewise, if $z^c > \tilde{z}$, we have $\pi_2^*(z^c) > \pi_1^*(z^c) \Longrightarrow \pi_2^*(z^c) = \pi^*(z^c)$, hence $z_2^m > 1$ is a unique global maximizer of $\pi(z)$. This proves part ii), which completes the proof. ■

Proposition A.

i) If $L/F < 1/\pi^*(\tilde{z})$, then in equilibrium all monopolists set the price $p_1^m = \psi/(1 - 1/\sigma)$

ii) If $L/F = 1/\pi^*(\tilde{z})$, then there is a continuum of equilibria;

iii) If $L/F > 1/\pi^*(\tilde{z})$, then in equilibrium all monopolists set the price $p_2^m = \psi/[1 - 1/(\sigma - \varepsilon)]$.

Proof. The zero-profit condition can be stated as follows:

$$\pi^*(z^c) = \frac{F}{L}.$$

Because $\pi^*(z^c) = \max\{\pi_1^*(z^c), \pi_2^*(z^c)\}$, $\pi^*(z^c)$ is a decreasing function, because it is an upper envelope of two decreasing functions. Combining this with Lemma 2A, we have:

$L/F < 1/\pi^*(\tilde{z}) \Leftrightarrow z^c < \tilde{z} \Leftrightarrow z_1^m < 1$ is the unique global profit maximizer;

$L/F = 1/\pi^*(\tilde{z}) \Leftrightarrow z^c = \tilde{z} \Leftrightarrow z_1^m < 1$ and $z_2^m > 1$ are global profit maximizers;

$L/F > 1/\pi^*(\tilde{z}) \Leftrightarrow z^c > \tilde{z} \Leftrightarrow z_2^m > 1$ is the unique global profit maximizer.

Observing that $p^m = \psi\, z^m/z^c$ completes the proof. ■

We now come to studying the behaviour of θ. When $L/F < 1/\pi^*(\tilde{z})$, we have:

$$\theta = \left(1 - \frac{1}{\sigma}\right)^{1-\sigma}.$$

However, as L/F reaches the level of $1/\pi^*(\tilde{z})$, θ jumps upwards and becomes:

$$\frac{\sigma}{\sigma - \varepsilon}\left(1 - \frac{1}{\sigma}\right)^{1-\sigma}.$$

Observe that this value is not bounded from above by e, and can be made arbitrarily large.

When $L/F \in (1/\pi^*(\tilde{z}), 1/\pi^*(1))$, we have:

$$\theta = \left(1 - \frac{1}{\sigma - \varepsilon}\right)^{1-(\sigma-\varepsilon)} (z^c)^{-\varepsilon}.$$

Because $z^c = (\pi^*)^{-1}(F/L)$ increases with L/F, while θ decreases with z^c, we conclude that θ decreases with L/F over $(1/\pi^*(\tilde{z}), 1/\pi^*(1))$. Finally, when $L/F \geq 1/\pi^*(1)$, θ is again constant and is given by:

$$\theta = \left(1 - \frac{1}{\sigma - \varepsilon}\right)^{1-(\sigma-\varepsilon)}.$$

To sum up, the impact of a growing L/F is, first, destabilizing, and then stabilizing.

Appendix D: (A2) alone does not ensure that θ is increasing in L/F.

The next example satisfies (A2), but not the log-concavity condition. And θ can be decreasing in L/F. Thus, (A2) alone is not sufficient for θ to be increasing in L/F.

Example D: Additively Perturbed CES

$$s(z) = \max\{z^{1-\sigma} - \varepsilon^{\sigma-1}, 0\}, (\sigma > 1;\ \varepsilon > 0) \Longrightarrow$$

$$\zeta(z) = \frac{\sigma z^{1-\sigma} - \varepsilon^{\sigma-1}}{z^{1-\sigma} - \varepsilon^{\sigma-1}} = \frac{\sigma - (\varepsilon z)^{\sigma-1}}{1 - (\varepsilon z)^{\sigma-1}} = \sigma + \frac{\sigma - 1}{(\varepsilon z)^{1-\sigma} - 1},$$

with $\sigma > 1$ and $\varepsilon > 0$. $\zeta(z)$ is strictly increasing in $z \in (0, 1/\varepsilon)$ with the range from σ to ∞. It thus satisfies (A2), but not the log-concavity. Hence, it is necessary to go through calculation explicitly as follows:

$$\theta = \frac{s\left(\underline{z}^m\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]\right)}{s(\underline{z}^m)} = \frac{\left(\underline{z}^m\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]\right)^{1-\sigma} - \varepsilon^{\sigma-1}}{(\underline{z}^m)^{1-\sigma} - \varepsilon^{\sigma-1}} = \frac{\left[\frac{\sigma - (\varepsilon \underline{z}^m)^{\sigma-1}}{\sigma - 1}\right]^{\sigma-1} - (\varepsilon \underline{z}^m)^{\sigma-1}}{1 - (\varepsilon \underline{z}^m)^{\sigma-1}},$$

where $\underline{z}^m$ is given by

$$\frac{L}{F} = \frac{\zeta(\underline{z}^m)}{s(\underline{z}^m)} = \frac{(\underline{z}^m)^{\sigma-1}}{1 - (\varepsilon \underline{z}^m)^{\sigma-1}} \frac{\sigma - (\varepsilon \underline{z}^m)^{\sigma-1}}{1 - (\varepsilon \underline{z}^m)^{\sigma-1}}$$

and it is strictly increasing in $L/F \in (0, \infty)$ with the range, $(0, 1/\varepsilon)$. Therefore, as $\underline{z}^m \to 0$, $\zeta(\underline{z}^m) \to \sigma$ and $\theta \to \left(1 - \frac{1}{\sigma}\right)^{1-\sigma}$ and, as $\underline{z}^m \to 1/\varepsilon$, $\zeta(\underline{z}^m) \to \infty$ and $\theta \to 2$. This means that, for $\sigma < 2$, θ is strictly increasing in $\underline{z}^m \in (0, 1/\varepsilon)$, hence strictly increasing in $L/F \in (0, \infty)$; for $\sigma = 2$, $\theta = 2$; and for $\sigma > 2$, θ is strictly decreasing in $\underline{z}^m \in (0, 1/\varepsilon)$, hence strictly decreasing in $L/F \in (0, \infty)$.

This example thus shows that, in spite of (A2), θ can be strictly decreasing in L/F.

Appendix E: Two Families of Perturbed CES satisfying the log-concavity and (A2) condition that jointly ensure that θ is increasing in L/F.

Example E1: Multiplicatively Perturbed CES with a Linear Elasticity Function

$$s(z) = (z\exp(\varepsilon z))^{1-\sigma} \Longrightarrow \zeta(z) = \sigma + (\sigma - 1)\varepsilon z,$$

where $\sigma > 1$ and $\varepsilon > 0$. Since $\zeta(z) - 1 = (\sigma - 1)(1 + \varepsilon z)$ is strictly increasing and strictly log-concave, one could immediately conclude from Proposition 1 that θ is bounded by e, and from the corollary of Proposition 2 that θ is strictly increasing in L/F. More explicitly,

$$\theta = \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \left(\frac{\underline{z}^c}{\underline{z}^m}\exp\left(\varepsilon(\underline{z}^c - \underline{z}^m)\right)\right)^{1-\sigma} = \left(\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]\exp\left(-\frac{\varepsilon\underline{z}^m}{\zeta(\underline{z}^m)}\right)\right)^{1-\sigma}$$

$$= \left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]^{1-\sigma}\exp\left(1 - \frac{\sigma}{\zeta(\underline{z}^m)}\right),$$

where $\underline{z}^m$ is given by

$$\frac{L}{F} = \frac{\zeta(\underline{z}^m)}{s(\underline{z}^m)} = [\sigma + (\sigma - 1)\varepsilon\underline{z}^m]\left(\underline{z}^m\exp(\varepsilon\underline{z}^m)\right)^{\sigma-1},$$

and it is strictly increasing in $L/F \in (0, \infty)$, with the range, $\underline{z}^m \in (0, \infty)$. Therefore, as $\underline{z}^m \to 0$, $\zeta(\underline{z}^m) \to \sigma$ and $\theta \to \left(1 - \frac{1}{\sigma}\right)^{1-\sigma} < e$, and as $\underline{z}^m \to \infty$, $\zeta(\underline{z}^m) \to \infty$ and $\theta \to e$.

Recall that the steady state is unstable, when $\delta(\theta - 1) > 1$. For $\sigma \geq 2, \theta > 2$ always holds. Hence, the steady state is unstable for a sufficiently high δ (i.e., sufficiently close to one). For $\sigma < 2$, there exists a critical value of L/F, at which $\theta = 2$. For L/F below this critical value, $\theta < 2$ and hence the steady state is always stable. For L/F above this critical value, $\theta > 2$ and hence the steady state is unstable for a sufficiently high δ (i.e., sufficiently close to one).

Example E2: Multiplicatively Perturbed CES with a Linear Fractional Elasticity Function

$$s(z) = z^{1-\sigma}(1 + z)^{-\varepsilon} \Longrightarrow \zeta(z) = \sigma + \frac{\varepsilon z}{1 + z}.$$

Again, $\sigma > 1$ and $\varepsilon > 0$. Since $\zeta(z)$ is strictly increasing and strictly concave, one could immediately conclude from Proposition 1 that θ is bounded by e, and from the corollary of Proposition 2 that θ is strictly increasing in L/F. More explicitly,

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \left(\frac{\underline{z}^c}{\underline{z}^m}\right)^{1-\sigma}\left(\frac{1+\underline{z}^c}{1+\underline{z}^m}\right)^{-\varepsilon} = \left(1-\frac{1}{\zeta(\underline{z}^m)}\right)^{1-\sigma}\left(\frac{1+\underline{z}^m\left(1-\frac{1}{\zeta(\underline{z}^m)}\right)}{1+\underline{z}^m}\right)^{-\varepsilon}$$

$$= \left(1-\frac{1}{\zeta(\underline{z}^m)}\right)^{1-\sigma}\left(1-\frac{1}{\zeta(\underline{z}^m)}\frac{\underline{z}^m}{1+\underline{z}^m}\right)^{-\varepsilon},$$

where $\underline{z}^m$ is given by

$$\frac{L}{F} = \frac{\zeta(\underline{z}^m)}{s(\underline{z}^m)} = \underline{z}^{m\sigma-1}(1+\underline{z}^m)^{\varepsilon}\left(\sigma+\frac{\varepsilon\underline{z}^m}{1+\underline{z}^m}\right),$$

and it is strictly increasing in $L/F \in (0,\infty)$, with the range, $\underline{z}^m \in (0,\infty)$. Therefore, as $\underline{z}^m \to 0$, $\zeta(\underline{z}^m) \to \sigma$ and $\theta \to \left(1-\frac{1}{\sigma}\right)^{1-\sigma} < e$ and, as $\underline{z}^m \to \infty$, $\zeta(\underline{z}^m) \to \sigma+\varepsilon$ and $\theta \to \left(1-\frac{1}{\sigma+\varepsilon}\right)^{1-\sigma-\varepsilon} < e$.

Recall that the steady state in unstable, when $\delta(\theta-1) > 1$. For $\sigma \geq 2, \theta > 2$ always holds. Hence, the steady state is unstable for a sufficiently high δ (i.e., sufficiently close to one). For $\sigma < 2 < \sigma+\varepsilon$, there exists a critical value of L/F, at which $\theta = 2$. For L/F below this critical value, $\theta < 2$ and hence the steady state is always stable. For L/F above this critical value, $\theta > 2$ and hence the steady state is unstable for a sufficiently high δ (i.e., sufficiently close to one). For $\sigma+\varepsilon < 2$, the steady state is stable.